# Final Exam - Math 217

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Name: ________________________________

Please circle your answers. Cross out any work that you do not want graded.

| **Problem** | **Grade** | **Max** |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 15 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 15 |
| 8 | | 10 |
| 9 | | 10 |
| Total | | 100 |

**Credit for problems 1-4 is based on your answers only.**

(1) Mark the following statements as either "True" or "False."

- If $A$ and $B$ are invertible square matrices, then $(AB)^{-1}$ must be equal to $A^{-1}B^{-1}$.

- Suppose that $S$ is an orthonormal set of vectors in $\mathbb{R}^5$. Then $S$ cannot contain more than 5 elements.

- There does not exist a linear transformation $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ that is both one-to-one and onto.

- The determinant of a diagonal matrix is always equal to the product of its diagonal entries.

- Suppose that $V$ is a vector space and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $V$. Then the coordinate mapping $V \to \mathbb{R}^n$ (given by $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$) must be an isomorphism.

(2) Mark the following statements as either "True" or "False."

- If $A$ is an $m \times n$ matrix, and $\mathbf{b}$ is a vector in $\mathbb{R}^m$, then the set of solutions to the equation $A\mathbf{x} = \mathbf{b}$ must be a subspace of $\mathbb{R}^n$.

- For any $n \times n$ matrix $A$ and any real number $c$,
$$\det(cA) = c^n \det(A).$$

- Let $\mathbf{w}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. If $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then the length of $(\mathbf{w} + \mathbf{v})$ must be the same as the length of $\mathbf{w}$.

- Let $M_{n\times n}$ denote the vector space of $n \times n$ matrices (with the usual rules for addition and scalar multiplication). Then
$$\dim M_{n\times n} = n$$
for any $n$.

- Let $A$ be a $n \times n$ matrix that has one (and only one) real eigenvalue. Then $A$ must be equal to a scalar multiple of the identity matrix.

(3) Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Compute $A^{-1}$.

(b) Compute $\det A$.

(4) Please circle your answers for the following problems.

(a) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which flips vectors across the origin. Find the standard matrix for $S$.

(b) Let
$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$
Find a vector $\mathbf{w}$ of length 1 which is orthogonal to $\mathbf{v}$.

(c) Let $T$ be a triangle in $\mathbb{R}^2$ whose vertices all have integer coordinates. What is the smallest possible area of the interior of $T$? (Assume that the vertices of $T$ do not all line on a single line.)

**Credit for problems 5-8 is based on answers and on work shown.**

(5) Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 0 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ -4 \end{bmatrix}.$$

Find a vector $\mathbf{x}$ which satisfies the equation $A\mathbf{x} = \mathbf{b}$.

(6) Let

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Compute the dimension of $H$. (Be sure to show work that supports your answer.)

(7) Let

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 7 \end{bmatrix}.$$

(a) Find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$.

(b) Let $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and consider the sequence of vectors

$$\mathbf{v}, A\mathbf{v}, A^2\mathbf{v}, \ldots$$

Which of the following statements best describes the geometric behavior of this sequence? (Circle one.)

- The sequence tends towards the origin.
- The sequence tends towards the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- The sequence tends towards the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- The sequence tends away from the origin.

(8) Let

$$H = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } \mathbf{v} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}.$$

Find the vector $\mathbf{w} \in H$ which makes the distance $\|\mathbf{w} - \mathbf{v}\|$ as small as possible.

(9) Let $A$ be a $10 \times 10$ matrix such that $A^2$ is the zero matrix. Prove that the dimension of Nul $A$ must be at least 5.