Problem

Let $S_1, S_2, \ldots, S_n$ be a nonincreasing sequence of positive integers from the interval $[1, n]$. Suppose that $S_{S_k} \geq k$ for all $k \in [1, n]$. Show that for some m,

$$\sum_{k=1}^{m} S_k - \sum_{k=m+1}^{n} S_k = m^2.$$

(Math Problem of the Week, 9/8/96)
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