Problem

Function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) + f''(x)| \le 1$ for all x. Given that $f(0) = f'(0) = 0$, show that $|f(x)| \le x$ for all $x \ge 0$.

Solution: Let

$$g(x) = \sqrt{f(x)^2 + f'(x)^2}.$$

Then, whenever $\sqrt{f(x)^2 + f'(x)^2} \neq 0$,

$$\begin{aligned} g'(x) &= \frac{\frac{d}{dx}\left(f(x)^2 + f'(x)^2\right)}{2\sqrt{f(x)^2 + f'(x)^2}} \\ &= \frac{2f(x)f'(x) + 2f'(x)f''(x)}{2\sqrt{f(x)^2 + f'(x)^2}} \\ &= (f(x) + f''(x))\left(\frac{f'(x)}{\sqrt{f(x)^2 + f'(x)^2}}\right). \end{aligned}$$

By assumption, $(f(x) + f''(x))$ lies between -1 and 1. Additionally, the quotient $f'(x)/\sqrt{f(x)^2 + f'(x)^2}$ must lie between -1 and 1. (For any real numbers a and b, $|b| \le \sqrt{a^2 + b^2}$.) Therefore, $-1 \le g'(x) \le 1$ whenever $g(x) \neq 0$.

We would like to show that $|f(x)| \le x$ for all $x > 0$. Since

$$|f(x)| \le \sqrt{f(x)^2 + f'(x)^2} = g(x),$$

it suffices to prove that $g(x) \le x$ for all $x > 0$.

Note that $g(0) = \sqrt{f(0)^2 + f'(0)^2} = 0$. Suppose that there exists some positive value x_0 which makes $g(x_0)$ greater than x_0. The inequality $g'(x) \le 1$ implies that the graph of $g(x)$ must lie above the line segment

$$y = x + (g(x_0) - x_0), \qquad 0 \le x \le x_0.$$

In particular, $g(0) \ge g(x_0) - x_0 > 0$, which is a contradiction. Therefore, $g(x) \le x$ and $|f(x)| \le x$ for all $x \ge 0$. $\square$