Problem

For any real polynomial $P(x) = p_0 + p_1 x + p_2 x^2 + \ldots$, define the expressions $\sin(P(x))$, $\cos(P(x))$, and $\tan(P(x))$ by

$$\sin(P(x)) = p_1 - p_3 + p_5 - p_7 + \ldots,$$

$$\cos(P(x)) = p_0 - p_2 + p_4 - p_6 + \ldots,$$

$$\tan(P(x)) = \frac{p_1 - p_3 + p_5 - p_7 + \ldots}{p_0 - p_2 + p_4 - p_6 + \ldots}.$$

Prove the identities:

$$\sin(P(x)Q(x)) = \sin(P(x))\cos(Q(x)) + \cos(P(x))\sin(Q(x)),$$

$$\cos(P(x)Q(x)) = \cos(P(x))\cos(Q(x)) - \sin(P(x))\sin(Q(x)),$$

$$\tan(P(x)Q(x)) = \frac{\tan(P(x)) + \tan(Q(x)))}{1 - \tan(P(x))\tan(Q(x))}.$$

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