Cohomology of p-torsion sheaves on characteristic-p curves

by

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Abstract

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This thesis studies the relationship between the category of constructible p-torsion étale sheaves and the category of quasi-coherent $\mathcal{O}_{F,X}$-modules. The main result is a bound on the Euler characteristic of a p-torsion étale sheaf on a characteristic-p curve. This result follows work of R. Pink.

Professor Arthur Ogus
Dissertation Committee Chair

"I don't think I ever deal with things that are abstract. ... If one is really thinking about [an] 'abstracted' concept and working with it seriously, it will become utterly as concrete as any other concept. Of course, one may have to homegrow the appropriate intuitions to deal seriously with it."

– Barry Mazur in an interview

"Space and Time! now I see it is true, what I guess'd at,
What I guess'd when I loaf'd on the grass,
What I guess'd while I lay alone in my bed,
And again as I walk'd the beach under the paling stars of the morning.
My ties and ballasts leave me."

– Walt Whitman, *Leaves of Grass*

Contents

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Chapter 1

Introduction

1.1 Étale cohomology on a characteristic-p curve

Let k_0 be a field of characteristic $p > 0$. If one is interested in the geometry of schemes over k_0, the class of étale sheaves is a useful tool. The étale cohomology groups of a k_0-scheme capture information about finite covers of the scheme and (if k_0 is finite) information about its zeta function. Étale sheaves tend to have familiar geometric properties which make them easy to manipulate. The literature on étale cohomology (such as Grothendieck's "SGA" series) provides many results for étale sheaves which parallel results for sheaves from classical geometry (such as local systems).

However this parallel is somewhat limited, since many of the existing results for étale sheaves do not apply in full generality. A theorem on étale sheaves of R-modules (where R is a finite ring) typically begins with the assumption that the order of R is relatively prime to the characteristic of k_0. Many results fail without this assumption. It seems that $\mathbb{F}_p$- and $\mathbb{Z}/p^n\mathbb{Z}$-étale sheaves are less tractable than $\mathbb{F}_\ell$- and $\mathbb{Z}/\ell^n\mathbb{Z}$-étale sheaves.

(Here ℓ denotes a prime different from p.)

Consider, for example, the theorem known as the "Grothendieck-Ogg-Shafarevich formula." Let k be the algebraic closure of k_0. If X is a smooth projective k-curve and N is a constructible $\mathbb{F}_\ell$-étale sheaf on X, this theorem computes the Euler characteristic of N in terms of local invariants.

Theorem 1.1.1 (Grothendieck, Ogg, Shafarevich) *Let ℓ be a prime different from p. Let X be a smooth projective curve of genus g over k. Let N be a constructible étale sheaf of $\mathbb{F}_\ell$-modules on X with generic rank n. Then*

$$\chi(X, N) = (2 - 2g)n - \sum_{x \in X(k)} \mathfrak{S}(N_{(x)}). \tag{1.1.2}$$

In this formula, $N_{(x)}$ denotes the pullback of N via the morphism

$$\text{Spec } (\mathcal{O}_{X,\text{ét}})_x \to X, \tag{1.1.3}$$

and the symbol $\mathfrak{S}$ denotes a integral invariant that is computed from $N_{(x)}$. (The invariant $\mathfrak{S}$ is related to the Swan conductor.)

It is easy to see why the assumption that $\ell \neq p$ in Theorem 1.1.1 is necessary. For example, let $f : E \to \mathbb{P}^1$ be a morphism from an elliptic curve E that has exactly four ramified points, each of ramification index 2. The Euler characteristic of the pushfoward sheaf $N := f_* \underline{\mathbb{F}_p}$ on $\mathbb{P}^1$ can be 0 or 1, depending on whether the elliptic curve E is supersingular (see Section 5.5). Thus the Euler characteristic of N cannot be computed from purely local data.

Despite their peculiarities, cohomology groups of $\mathbb{F}_p$-étale sheaves carry some important information. Zeta functions with $\mathbb{F}_p$-coefficients can be computed from $\mathbb{F}_p$-étale cohomology groups (see Exposé XXII in [3]). If X is a smooth projective k-curve as above, the dimension of $H^1(X, \mathbb{F}_p)$ determines the size of the kernel of the multiplication-by-p map

$$\cdot^{\otimes p} \colon \mathrm{Jac}(X) \to \mathrm{Jac}(X) \tag{1.1.4}$$

on the Jacobian of X. The size of this kernel is essential for studying deformations of $\mathrm{Jac}(X)$.

Let M be a constructible $\mathbb{F}_p$-étale sheaf on the curve X. Although the Euler characteristic of M cannot be determined from local data, partial information can sometimes be obtained. If M is locally constant, then

$$\chi(X, M) \geq (1-g)n, \tag{1.1.5}$$

where n is the generic rank of M. If $M = f_* \underline{\mathbb{F}_p}$, where $f \colon Y \to X$ is a finite morphism of smooth curves, then the Riemann-Hurwitz formula can be used to determine a lower bound for $\chi(X, M)$ in terms of the ramification of f.

In *Open Problems in Algebraic Geometry* ([5]), Richard Pink suggested constructing a general lower bound for the Euler characteristic of a constructible $\mathbb{F}_p$-étale sheaf on a characteristic-p curve. Pink proposed finding a bound that depends (like the Grothendieck-Ogg-Shafarevich formula) only on the genus of the curve, the generic rank of the sheaf, and local data for the sheaf. The main purpose of this thesis is to present such a bound.

Theorem 1.1.6 *Let X be a smooth projective curve of genus g over k. Let M be a constructible $\mathbb{F}_p$-étale sheaf on X. Let n be the generic rank of M. Then*

$$\chi(X, M) \geq (1-g)n - \sum_{x \in X(k)} \mathfrak{C}(M_{(x)}). \tag{1.1.7}$$

The local term $\mathfrak{C}(M_{(x)})$ (defined in Section 5.4) is called the *minimal root index* of M at x. The proof of this theorem is the result of a study of the relationships between $\mathbb{F}_p$-étale sheaves and $\mathcal{O}_X$-modules.

1.2 Étale sheaves and quasi-coherent $\mathcal{O}_X$-modules

The étale structure sheaf of a k-scheme is itself a p-torsion étale sheaf. This fact provides unique methods for investigating $\mathbb{F}_p$-étale sheaves. We continue with notation from the previous section. Let M be a constructible $\mathbb{F}_p$-étale sheaf on the curve X. Since M is constructible, it is locally constant on some nonempty open subset U of X. The sheaf $\mathcal{P} := (M_{|U}) \otimes_{\mathbb{F}_p} \mathcal{O}_U$ is therefore a locally-free $\mathcal{O}_U$-module of rank n. The Frobenius endomorphism $[f \mapsto f^p]$ of $\mathcal{O}_U$ determines a Frobenius-linear endomorphism of $\mathcal{P}$, thus making $\mathcal{P}$ an F-crystal on U. One can also construct a dual sheaf,

$$\mathcal{P}^\vee := \mathcal{H}om_{\mathbb{F}_p}(M_{|U}, \mathcal{O}_U). \tag{1.2.1}$$

The sheaf $\mathcal{P}^\vee$ also has the structure of an F-crystal on U. This dual association has the advantage that it extends canonically to the entire curve X. Let

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_p}(M, \mathcal{O}_X). \tag{1.2.2}$$

Then $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module with a Frobenius-linear endomorphism (see Proposition 5.1.18).

The sheaf M can be recovered (except for sections with punctual support) from the data of the sheaf $\mathcal{M}$ and its endomorphism. This association is part of the characteristic-p "Riemann-Hilbert" correspondence of M. Emerton and M. Kisin. In [1], these authors define an anti-equivalence of categories which relates $\mathbb{F}_p$-étale sheaves on a smooth k-scheme to quasi-coherent modules with Frobenius-linear endomorphisms. (This is an analogue of the characteristic-0 Riemannn-Hilbert correspondence, which relates local systems on complex manifolds to vector bundles with integrable connection.) The correspondence of Emerton and Kisin includes relationships between derived pushforward functors, which imply in particular a relationship between the cohomologies of the sheaves M and $\mathcal{M}$ above.

In this paper we borrow a number of concepts from [1], including in particular the notion of a "root".

Definition 1.2.3 *Let $\mathcal{N}$ be a quasi-coherent $\mathcal{O}_X$-module with a Frobenius linear endomorphism $\phi\colon \mathcal{N} \to \mathcal{N}$. A coherent submodule $\mathcal{N}' \subseteq \mathcal{N}$ is a* root *of $\mathcal{N}$ if*

- *$\{\phi^n(\mathcal{N}')\}_{n=0}^{\infty}$ generates $\mathcal{N}$ as an $\mathcal{O}_X$-module, and*
- *The $\mathcal{O}_X$-submodule of $\mathcal{N}$ generated by $\phi(\mathcal{N}')$ contains $\mathcal{N}'$.*

Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root for $\mathcal{M}$. The coherent $\mathcal{O}_X$-module $\mathcal{M}_0$ is a sheaf of morphisms from M into $\mathcal{O}_X$, and thus there exists a double-dual homomorphism

$$M \to \mathcal{M}_0^\vee = \mathcal{H}om(\mathcal{M}_0, \mathcal{O}_X). \tag{1.2.4}$$

This morphism determines homomorphisms

$$H^j(X, M) \otimes_{\mathbb{F}_p} k \to H^j(X, \mathcal{M}_0^\vee) \tag{1.2.5}$$

for each $j \geq 0$. We will show that homomorphism (1.2.5) is bijective when $j = 0$ and injective when $j = 1$. As a consequence, the Euler characteristic of M is bounded below by the Euler characteristic of $\mathcal{M}_0^\vee$. This observation provides the essential step for the proof

Theorem 1.1.6. Since $\mathcal{M}_0^\vee$ is coherent, its Euler characteristic is determined by its degree. We prove the existence of a canonical minimal root $\mathcal{M}_0 \subseteq \mathcal{M}$ whose degree depends on local data for the sheaf M.

This approach is similar to the one that R. Pink used in [6] to prove a partial answer to his aforementioned proposal. Pink proved a lower bound on the Euler characteristic of M under the assumption that M has no higher wild ramification at any closed point of X (Proposition 7.1 in [6]). In this paper, the notion of a root allows us to manage sheaves with arbitrary wild ramification. The key new construction in this paper is the construction of the canonical minimal root of $\mathcal{M}$ (Proposition 5.3.24).

1.3 Outline of thesis

Chapter 2 reviews basic properties of étale sheaves of abelian groups. The localization functor $N \mapsto N_{(x)}$ is defined. The notion of constructibility is introduced. If N is a constructible sheaf of abelian groups on a smooth projective curve, then the cohomology groups of N are finite (Proposition 2.2.3).

Chapter 3 introduces the sheaf $\mathcal{O}_{F,X}$ and the class of left $\mathcal{O}_{F,X}$-modules. The notation in this chapter is borrowed from [1]. The sheaf $\mathcal{O}_{F,X}$ on a scheme X is the sheaf obtained by formally adjoining to $\mathcal{O}_X$ an element representing the Frobenius endomorphism. The introduction of the sheaf $\mathcal{O}_{F,X}$ allows one to deal more easily with the $\mathcal{O}_X$-modules that arise from $\mathbb{F}_p$-étale sheaves. An $\mathcal{O}_X$-module with a Frobenius-linear endomorphism is a left $\mathcal{O}_{F,X}$-module. The left $\mathcal{O}_{F,X}$-modules that arise from constructible $\mathbb{F}_p$-étale sheaves (such as $\mathcal{M}$ in the previous section) are of a special type which we will call *finite-unit* $\mathcal{O}_{F,X}$-

modules. The notion of a root of an $\mathcal{O}_{F,X}$-module is introduced in Section 3.3. We quote a theorem from [1] that asserts that every finite-unit $\mathcal{O}_{F,X}$-module has a root.

Chapter 4 is devoted to the study of modules over a one-dimensional Henselian regular local ring A. This chapter performs the necessary work for the definition of the local terms in Theorem 1.1.6. We denote by $A[F]$ the twisted polynomial algebra determined by the Frobenius endomorphism of A. We define the *F-index* of an A-submodule of a left $A[F]$-module. This invariant measures how far the submodule is moved by the Frobenius endomorphism. We consider the set of roots in a finite-unit $A[F^r]$-module and prove that there must exist one root which is minimal under inclusion. The *minimal root index* of a finite-unit $A[F^r]$-module is the F^r-index of its smallest root. Finally, we prove a local version (Theorem 4.4.15) of the characteristic-p Riemann-Hilbert correspondence, which asserts the existence of a correspondence between finite-unit $A[F]$-modules and constructible $\mathbb{F}_p$-étale sheaves on Spec A.

Chapter 5 proves the characteristic-p Riemann-Hilbert correspondence for a smooth curve X. The basic method of proof is reduction to the local correspondence. If M is a constructible $\mathbb{F}_p$-étale sheaf on a curve and $\mathcal{M}$ is its associated finite-unit $\mathcal{O}_{F,X}$-module, then the localizations $\mathcal{M}_x$ and $M_{(x)}$ are related by the local Riemann-Hilbert correspondence. Localization also allows the construction of the canonical minimal root $\mathcal{M}_0 \subseteq \mathcal{M}$. The degree of $\mathcal{M}_0$ as a coherent sheaf is the sum of the local minimal root indicies of $\mathcal{M}$. Finally, the main result (Theorem 5.4.9) is proved via the Riemann-Roch Theorem.

1.4 Notation and conventions

Throughout this paper, p denotes a positive prime and k denotes an algebraically closed field of characteristic p. All schemes are assumed to be noetherian and separated. The term "k-curve" is used to mean irreducible k-curve. All sheaves are assumed to be sheaves on an étale site. If X is a k-scheme, then $\mathcal{O}_X$ denotes the étale structure sheaf.

If $\mathcal{R}$ is a sheaf of rings on a k-scheme X, let $\mathbf{LMod}(X, \mathcal{R})$ and $\mathbf{RMod}(X, \mathcal{R})$ (or simply $\mathbf{Mod}(X, \mathcal{R})$ if $\mathcal{R}$ is commutative) denote the categories of left and right $\mathcal{R}$-modules, respectively. Cohomology functors for $\mathbf{LMod}(X, \mathcal{R})$ are calculated in the category $\mathbf{Mod}(X, \mathbb{Z})$ of étale sheaves of abelian groups. If $\mathcal{F}$ is an $\mathcal{R}$-module, then $H^j(X, \mathcal{F})$ inherits a $\Gamma(X, \mathcal{R})$-module structure for each $j \geq 0$.

If R is a k-algebra, let $F_R \colon R \to R$ denote the Frobenius endomorphism $[r \mapsto r^p]$ for R. If X is a k-scheme, let $F_X \colon X \to X$ denote the Frobenius endomorphism for X.

Chapter 2

Étale sheaves of abelian groups

This chapter reviews some basic operations for étale sheaves of abelian groups. The primary reference for this material is [2].

2.1 Stalks, pushforwards, and pullbacks

Let Y be a k-scheme, and let $\mathcal{F}$ be an object in $\mathbf{Mod}(Y, \mathbb{Z})$. If y is a point of Y, and

$$s\colon \operatorname{Spec} \overline{k(y)} \to Y \tag{2.1.1}$$

is a geometric point of Y at y, then $\mathcal{F}_s$ denotes the stalk of $\mathcal{F}$ at s. (If $k(y) = k$, then we may write $\mathcal{F}_y$ for $\mathcal{F}_s$.)

Let $f\colon Y \to Z$ be a morphism of schemes over k. Then

$$f_*\colon \mathbf{Mod}(Y, \mathbb{Z}) \to \mathbf{Mod}(Z, \mathbb{Z}) \tag{2.1.2}$$

and

$$f^{-1}\colon \mathbf{Mod}(Z,\mathbb{Z}) \to \mathbf{Mod}(Y,\mathbb{Z}) \tag{2.1.3}$$

denote the pushforward and pullback functors associated to f. The functor f_* is defined by

$$(f_*\mathcal{G})\,(U) = \mathcal{G}(U \times_Z Y) \tag{2.1.4}$$

for any $\mathcal{G} \in \mathrm{Ob}\,\mathbf{Mod}(Y,\mathbb{Z})$ and any étale morphism $U \to Z$. The functor f^{-1} is a left adjoint to the functor f_*. If $g\colon Z \to V$ is another morphism of k-schemes, there is a canonical isomorphism between the functors $g_* \circ f_*$ and $(g \circ f)_*$, and a canonical isomorphism between the functors $f^{-1} \circ g^{-1}$ and $(g \circ f)^{-1}$.

Let

$$s\colon \mathrm{Spec}\,\overline{k(z)} \to Z \tag{2.1.5}$$

be a geometric point of Z. Then the composition of the pullback functor s^{-1} with the global sections functor $\Gamma(\cdot, \mathrm{Spec}\,\overline{k(z)})$ is canonically isomorphic to the stalk functor $(\cdot)_s$. (The global sections functor $\Gamma(\cdot, \mathrm{Spec}\,\overline{k(z)})$ is an isomorphism between $\mathbf{Mod}(\mathrm{Spec}\,\overline{k(z)}, \mathbb{Z})$ and the category of abelian groups.)

See sections I.2 and I.3 of [2] for a full discussion of the functors described above.[1] We quote here a few of the assertions from those sections.

Proposition 2.1.6 *Let X be a k-scheme. For any k-point x of X, the stalk functor $[\mathcal{G} \mapsto \mathcal{G}_x]$ from $\mathbf{Mod}(X,\mathbb{Z})$ to the category of abelian groups is exact.*

Proof. This is Remark I.2.12 in [2]. □

[1] The functor that we denote by f^{-1} is denoted by f^* in [2]. We use f^{-1} to avoid confusion with the $\mathcal{O}_Z$-module pullback functor.

Proposition 2.1.7 *Let $f\colon Y \to Z$ be a finite morphism of schemes over k, let z be a k-point of Z, and let $\mathcal{F} \in \mathrm{Ob}\ \mathbf{Mod}(Y, \mathbb{Z})$. The natural homomorphism*

$$(f_*\mathcal{F})_z \to \bigoplus_{f(y)=z} \mathcal{F}_y \tag{2.1.8}$$

(where the sum is taken over the k-points of Y that map to z) is an isomorphism.

Proof. This is a special case of Proposition I.3.3 in [2]. □

Let X be a k-scheme and let $j\colon U \hookrightarrow X$ be the inclusion of an open subscheme. Then

$$j_!\colon \mathbf{Mod}(U, \mathbb{Z}) \to \mathbf{Mod}(X, \mathbb{Z}) \tag{2.1.9}$$

denotes extension by zero. If $\mathcal{F} \in \mathrm{Ob}\ \mathbf{Mod}(U, \mathbb{Z})$, then

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in |U| \\ \{0\} & \text{if } x \notin |U| \end{cases} \tag{2.1.10}$$

for any k-point x of X.

Let Y be a k-scheme, let y be a k-point of Y, and let $\mathcal{G} \in \mathrm{Ob}\ \mathbf{Mod}(Y, \mathbb{Z})$. Then $\mathcal{G}_{(y)}$ denotes the pullback of $\mathcal{G}$ via the morphism

$$\mathrm{Spec}\ \mathcal{O}_{Y,y} \to Y. \tag{2.1.11}$$

(Here, $\mathcal{O}_{Y,y}$ denotes the stalk at y of the étale structure sheaf on Y.) The set of global sections of $\mathcal{G}_{(y)}$ is canonically isomorphic to the stalk $\mathcal{G}_y$.

2.2 Constructible étale sheaves on curves

We define the class of constructible étale sheaves for one-dimensional k-schemes. More general definitions can be found in Section I.4 of [2].

Definition 2.2.1 *Let X be a k-scheme whose dimension is less than or equal to* 1*. A sheaf* $\mathcal{F} \in \mathrm{Ob}\,\mathbf{Mod}(X, \mathbb{Z})$ *is* constructible *if its stalks are finite and it is locally constant on a dense Zariski open subset of X.*

If Y is a k-scheme and R is a finite commutative ring, then $\mathbf{Mod}^c(Y, R)$ denotes the full subcategory of constructible sheaves in $\mathbf{Mod}(Y, R)$. The category $\mathbf{Mod}^c(Y, R)$ is an abelian subcategory of $\mathbf{Mod}(Y, R)$ (see the Corollary of Proposition 4.8 in [2]).

Proposition 2.2.2 *Let $f\colon W \to X$ be a finite morphism of k-schemes whose dimensions are less than or equal to* 1*. Let $\mathcal{F}$ be a constructible étale sheaf of abelian groups on W. Then $f_*\mathcal{F}$ is constructible.*

Proof. This is a special case of Lemma I.4.11 in [2]. □

Proposition 2.2.3 *Let X be a smooth projective k-curve, and let $\mathcal{F}$ be a constructible étale sheaf of abelian groups on X. Then the cohomology groups $H^i(X, \mathcal{F})$ ($i \geq 0$) are finite.*

Proof. This is a special case of Theorem I.8.10 in [2]. □

Remark 2.2.4 Proposition 2.2.3 does not hold if we remove the assumption that X is projective. For example, the group $H^1(\mathbb{A}^1_k, \mathbb{F}_p)$ is infinite.

Proposition 2.2.5 *Let $f\colon X \to Y$ be a finite morphism of smooth projective curves over k, let R be a finite commutative ring, and let $\mathcal{F} \in \mathbf{Mod}^c(X, R)$. Then for any $j \geq 0$, the R-modules $H^j(Y, f_*\mathcal{F})$ and $H^j(X, \mathcal{F})$ are isomorphic.*

Proof. The functor

$$f_*\colon \mathbf{Mod}(X, \mathcal{F}) \to \mathbf{Mod}(Y, \mathcal{F}). \tag{2.2.6}$$

is exact, so its higher right-derived functors are trivial. The cohomology groups $H^j(X,\mathcal{F})$ may be computed via a Leray-Serre spectral sequence in which the second term is

$$\begin{array}{cccc} H^0(Y,f_*\mathcal{F}) & 0 & 0 & \cdots \\ H^1(Y,f_*\mathcal{F}) & 0 & 0 & \cdots \\ H^2(Y,f_*\mathcal{F}) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.2.7}$$

Thus $H^j(X,\mathcal{F})$ is isomorphic to $H^j(Y,f_*\mathcal{F})$ for all $j \geq 0$. $\square$

Chapter 3

$\mathcal{O}_{F^r,X}$-modules

This chapter is concerned with $\mathcal{O}_X$-modules that have Frobenius-linear endomorphisms. We borrow notation and terminology from [1].

Let r be a positive integer, and let $q = p^r$.

3.1 The Frobenius endomorphism of a k-scheme

Proposition 3.1.1 *Let X be a smooth k-scheme of dimension n. Then the rth Frobenius endomorphism $F_X^r\colon X \to X$ is finite and flat of degree q^n.*

Proof. Take any point $x \in |X|$. Since X is smooth of dimension n, there exists an affine neighborhood U of x and an étale morphism $f\colon U \to \mathbb{A}_k^n$. Consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{F^r} & U \\ \downarrow & & \downarrow \\ \mathbb{A}_k^n & \xrightarrow{F^r} & \mathbb{A}_k^n. \end{array} \qquad (3.1.2)$$

From the universal mapping property for the fibered product we obtain the diagram

$$\begin{array}{ccccc} & & \overset{F^r}{\frown} & & \\ U & \longrightarrow & U \times_{\mathbb{A}^n_k} \mathbb{A}^n_k & \longrightarrow & U \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{A}^n_k & \xrightarrow{F^r} & \mathbb{A}^n_k. \end{array} \tag{3.1.3}$$

The bottom morphism is finite and flat of degree q^n, and so the same is true of the morphism $U \times_{\mathbb{A}^n_k} \mathbb{A}^n_k \to U$. The morphism $U \to U \times_{\mathbb{A}^n_k} \mathbb{A}^n_k$ is an isomorphism (see Proposition 2 of Expose XV in [4]), and therefore F^r_U is finite and flat of degree q^n. □

Let R be a k-algebra. Then $R[F]$ is a ring whose elements are formal sums

$$\sum_{i=0}^{n} r_i F^i, \tag{3.1.4}$$

where n is a positive integer and $r_i \in R$. Multiplication in $R[F]$ is given by the rule $Fr = r^p F$. The ring $R[F^r]$ is the subring of $R[F]$ generated by R and F^r. Let X be a k-scheme. Then $\mathcal{O}_{F,X}$ is a sheaf of rings which contains $\mathcal{O}_X$ such that for any affine open subset $U = \text{Spec } R'$ of X,

$$\mathcal{O}_{F,X}(U) = R'[F]. \tag{3.1.5}$$

The sheaf $\mathcal{O}_{F^r,X}$ is the subsheaf of rings of $\mathcal{O}_{F,X}$ generated by $\mathcal{O}_X$ and F^r.

3.2 Left $\mathcal{O}_{F^r,X}$-modules

Let X be a k-scheme. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module, and let

$$\phi\colon F^{r*}_X \mathcal{M} \to \mathcal{M} \tag{3.2.1}$$

be an $\mathcal{O}_X$-linear morphism. Then a left $\mathcal{O}_{F^r,X}$-module structure on $\mathcal{M}$ is given by $F^r(m) = \phi(1 \otimes m)$ (where m denotes a section of $\mathcal{M}$). Conversely, a left $\mathcal{O}_{F^r,X}$-module structure on

$\mathcal{M}$ uniquely determines an $\mathcal{O}_X$-linear morphism $F_X^{r*}\mathcal{M} \to \mathcal{M}$. We refer to this morphism as the *structural morphism* of the left $\mathcal{O}_{F^r,X}$-module $\mathcal{M}$.

Definition 3.2.2 *Let X be a k-scheme. An object $\mathcal{M}$ of* $\mathbf{LMod}(\mathcal{O}_{F^r,X})$ *is a* unit $\mathcal{O}_{F^r,X}$-module *if $\mathcal{M}$ is quasi-coherent as an $\mathcal{O}_X$-module, and the structural morphism $F_X^*\mathcal{M} \to \mathcal{M}$ is an isomorphism. Let* $\mathbf{LMod}^u(\mathcal{O}_{F^r,X})$ *denote the full subcategory of* $\mathbf{LMod}(\mathcal{O}_{F^r,X})$ *consisting of unit $\mathcal{O}_{F^r,X}$-modules.*

Definition 3.2.3 *Let X be a k-scheme. A unit $\mathcal{O}_{F^r,X}$-module $\mathcal{M}$ is* finite-unit *if for every open affine subset U of X, the left $\mathcal{O}_X(U)[F^r]$-module $\mathcal{M}(U)$ is finitely-generated. Let* $\mathbf{LMod}^{fu}(\mathcal{O}_{F^r,X})$ *denote the full subcategory of* $\mathbf{LMod}(\mathcal{O}_{F^r,X})$ *consisting of finite-unit $\mathcal{O}_{F^r,X}$-modules.*

Remark 3.2.4 In [1], finite-unit $\mathcal{O}_{F^r,X}$-modules are referred to as "locally finitely-generated unit $\mathcal{O}_{F^r,X}$-modules."

Remark 3.2.5 When dealing with affine schemes it is often easiest to work directly with modules over rings. If R is a k-algebra, a left $R[F^r]$-module is a unit $R[F^r]$-module if and only if its structural morphism is an isomorphism. A unit $R[F^r]$-module is finite-unit if and only if it is finitely generated. The subcategories $\mathbf{LMod}^u(R[F^r])$ and $\mathbf{LMod}^{fu}(R[F^r])$ are defined as above. The categories $\mathbf{LMod}^u(R[F^r])$ and $\mathbf{LMod}^u(\mathcal{O}_{F^r,\mathrm{Spec}\ R})$ are equivalent, and the categories $\mathbf{LMod}^{fu}(R[F^r])$ and $\mathbf{LMod}^{fu}(\mathcal{O}_{F^r,\mathrm{Spec}\ R})$ are equivalent.

Let $f\colon X \to Y$ be a morphism of k-schemes. There is a natural morphism

$$\mathcal{O}_{F^r,Y} \to f_*\mathcal{O}_{F^r,X}. \tag{3.2.6}$$

If $\mathcal{M}$ is a left $\mathcal{O}_{F^r,X}$-module, then the pushforward $f_*\mathcal{M}$ has the structure of a left $\mathcal{O}_{F^r,Y}$-module (via (3.2.6)). If $\mathcal{N}$ is a left $\mathcal{O}_{F^r,Y}$-module, then the $\mathcal{O}_X$-module

$$f^*\mathcal{N} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N} \tag{3.2.7}$$

has the structure of a left $\mathcal{O}_{F^r,X}$-module given by $F^r(f \otimes n) = f^q \otimes F^r(n)$.

Proposition 3.2.8 *Let $f\colon X \to Y$ be a morphism of k-schemes. The functor*

$$f^*\colon \mathbf{LMod}(\mathcal{O}_{F^r,Y}) \to \mathbf{LMod}(\mathcal{O}_{F^r,X}) \tag{3.2.9}$$

restricts to a functor

$$f^*\colon \mathbf{LMod}^{fu}(\mathcal{O}_{F^r,Y}) \to \mathbf{LMod}^{fu}(\mathcal{O}_{F^r,X}). \tag{3.2.10}$$

Proof. Let $\mathcal{N}$ be a sheaf in $\mathbf{LMod}^{fu}(\mathcal{O}_{F^r,Y})$. The structural morphism for $f^*\mathcal{N}$ is the composite of two morphisms:

$$F_X^{r*} f^*\mathcal{N} \to f^* F_Y^{r*}\mathcal{N} \to f^*\mathcal{N}, \tag{3.2.11}$$

the first of which is determined by the diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X^r} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{F_Y^r} & Y, \end{array} \tag{3.2.12}$$

and the second of which is determined by the structural morphism of $\mathcal{N}$. Both of these morphisms are isomorphisms. Therefore $f^*\mathcal{N}$ is a unit $\mathcal{O}_{F^r,X}$-module.

Let $V \subseteq Y$ be an affine open subset. Let $\{n_1, \ldots, n_k\} \subseteq \mathcal{N}(V)$ be a subset that generates $\mathcal{N}(V)$ as a left $\mathcal{O}_{F^r,V}$-module. The sheaf $(f^*\mathcal{N})_{|f^{-1}(V)}$ is generated as a left $\mathcal{O}_{F^r,f^{-1}(V)}$-module by the pullbacks of the sections n_i. Therefore $f^*\mathcal{N}$ is finite-unit. $\square$

3.3 Roots of unit $\mathcal{O}_{F^r,X}$-modules

Let X be a k-scheme and let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module. As an $\mathcal{O}_X$-module, $\mathcal{M}$ is quasi-coherent but not necessarily coherent. In place of $\mathcal{M}$ itself it is often useful to consider a particular type of coherent submodule of $\mathcal{M}$.

Definition 3.3.1 *Let X be a k-scheme and let $\mathcal{M}$ be a unit $\mathcal{O}_{F^r,X}$-module. An $\mathcal{O}_X$-submodule $\mathcal{M}' \subseteq \mathcal{M}$ is a* root of $\mathcal{M}$ *if:*

1. *the $\mathcal{O}_X$-module $\mathcal{M}'$ is coherent,*
2. *the $\mathcal{O}_X$-submodule of $\mathcal{M}$ generated by $F^r(\mathcal{M}')$ contains $\mathcal{M}'$, and*
3. *as a left $\mathcal{O}_{F^r,X}$-module, $\mathcal{M}$ is generated by $\mathcal{M}'$.*

Remark 3.3.2 Our use of the term "root" is slightly different from the use of the term "root" in [1]. The authors of [1] would refer to the inclusion morphism $\mathcal{M}' \hookrightarrow \mathcal{M}$ as a root.

Proposition 3.3.3 *Let X be a smooth k-scheme and let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module. Then there exists a root $\mathcal{M}' \subseteq \mathcal{M}$.*

Proof. This is Theorem 6.1.3 in [1]. □

Proposition 3.3.4 *Let X be a smooth k-scheme. Let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module, and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root for $\mathcal{M}$. For each $n \geq 1$, let $\mathcal{M}_n$ be the $\mathcal{O}_X$-submodule of $\mathcal{M}$ generated by $F^{rn}(\mathcal{M}_0)$. For any $n \geq 0$, the structural morphism of $\mathcal{M}$ determines an isomorphism*

$$F_X^{rn*}\mathcal{M}_0 \xrightarrow{\cong} \mathcal{M}_n. \tag{3.3.5}$$

For any $n \geq 0$, $\mathcal{M}_{n+1}$ contains $\mathcal{M}_n$. The union of the set of submodules $\{\mathcal{M}_n\}_{n \geq 0}$ is $\mathcal{M}$.

Proof. Since $F_X^{rn\#}$ makes $\mathcal{O}_X$ a flat $\mathcal{O}_X$-module (see Proposition 3.1.1), the inclusion

$$\mathcal{M}_0 \hookrightarrow \mathcal{M} \tag{3.3.6}$$

determines an injection

$$F_X^{rn*}\mathcal{M}_0 \hookrightarrow F_X^{rn*}\mathcal{M}. \tag{3.3.7}$$

Composing this injection with the nth power of the structural morphism of $\mathcal{M}$ yields an injection

$$F_X^{rn*}\mathcal{M}_0 \hookrightarrow \mathcal{M} \tag{3.3.8}$$

whose image (by definition) is $\mathcal{M}_n$. The first assertion of the proposition is proved.

The assertion that $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$ follows from the observation that $\mathcal{M}_{n+1}$ is the sub-$\mathcal{O}_X$-module generated by $F^{rn}(\mathcal{M}_1)$. Since $\mathcal{M}_1$ contains $\mathcal{M}_0$ (by property 2 of Definition 3.3.1), $\mathcal{M}_{n+1}$ contains $\mathcal{M}_n$. The last assertion of the proposition follows from property 3 of Definition 3.3.1. □

Proposition 3.3.9 *Let X be a smooth k-scheme, let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module, and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root. Then there exists a dense open subset $U \subseteq X$ such that*

$$(\mathcal{M}_0)_{|U} = \mathcal{M}_{|U}. \tag{3.3.10}$$

Proof. It suffices to prove the assertion in the case in which X is connected (and therefore integral). Let $\mathcal{M}_1$ be the $\mathcal{O}_X$-module generated by $F^r(\mathcal{M}_0) \subseteq \mathcal{M}$. The coherent sheaf $\mathcal{M}_1$ contains $\mathcal{M}_0$, and isomorphism (3.3.5) implies that these two sheaves have the same generic rank. Therefore $\mathcal{M}_1/\mathcal{M}_0$ is supported at a proper closed subset of X. Let U be the complement of this closed subset. The subsheaf $(\mathcal{M}_0)_{|U} \subseteq \mathcal{M}_{|U}$ is stabilized by F^r. Since $(\mathcal{M}_0)_{|U}$ generates $\mathcal{M}_{|U}$ as an $\mathcal{O}_{F^r,U}$-module, it follows that $(\mathcal{M}_0)_{|U} = \mathcal{M}_{|U}$. □

Corollary 3.3.11 *Let X be a smooth k-scheme and let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module. Then there exists a dense open subset of X on which $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module.* □

3.4 Left $\mathcal{O}_{F^r,X}$-modules over a field

This section is concerned with left $L[F^r]$-modules, where L is a field. We will show that left $L[F^r]$-modules have a decomposition which resembles Jordan decomposition.

Proposition 3.4.1 *Let L be a perfect field that contains k, and let V be a left $L[F^r]$-module. For any n, the image of V under the action of F^{rn} is a sub-L-vector space of V. If V is an m-dimensional L-vector space, then the dimension of this image is m minus the dimension of the kernel of F^{rn} in V.*

Proof. Both assertions follow easily from the fact that the Frobenius endomorphisms of L are bijective. □

Definition 3.4.2 *Let L be a perfect field that contains k, and let V be a left $L[F^r]$-module. Then $V^{nil} \subseteq V$ is the subspace consisting of elements that are killed by F^{rn} for some $n \geq 0$, and $V^{surj} \subseteq V$ is the intersection of the subspaces $F^{rn}V \subseteq V$ for $n = 0, 1, 2, \ldots$.*

Proposition 3.4.3 *Let L be a perfect field that contains k, and let V be a left $L[F^r]$-module that is finite-dimensional over L. The subspace V^{nil} is killed by some power of F^r. The* *action of F^r on V^{surj} is bijective. The homomorphism*

$$V^{nil} \oplus V^{surj} \to V \tag{3.4.4}$$

is an isomorphism.

Proof. For any $n \geq 1$, let K_n and I_n denote the kernel and image, respectively, of the action of F^{rn} on V. The sequence of subspaces

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots \tag{3.4.5}$$

must stabilize since V is finite dimensional. The descending sequence

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \tag{3.4.6}$$

stabilizes simultaneously since $\dim_L K_n + \dim_L I_n = \dim_L V$ (Proposition 3.4.1). Choose $n_0 > 0$ such that $K_{n_0} = K_{n_0+1}$ and $I_{n_0} = I_{n_0+1}$. Then $V^{nil} = K_{n_0}$ and $V^{surj} = I_{n_0}$. The first two assertions of the proposition follow. For the third assertion, note that $V^{nil} \cap V^{surj} = \{0\}$. The homomorphism

$$V^{nil} \oplus V^{surj} \to V \tag{3.4.7}$$

is therefore injective. Since $\dim_L(V^{nil} \oplus V^{surj}) = \dim_L V$, this homomorphism is bijective. □

Proposition 3.4.8 *Let L be a perfect field that contains k. Let V be a left $L[F^r]$-module. Then V is finite-unit if and only if V is finite-dimensional over L and the action of F^r on V is bijective.*

Proof. Suppose that V is finite-unit. By Proposition 3.3.3, there exists a root $V_0 \subseteq V$. Let

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \tag{3.4.9}$$

be the filtration for V described in Proposition 3.3.4. Isomorphism (3.3.5) implies that $\dim V_n = \dim V_0$ for each $n \geq 1$. Therefore $V = V_0$ and thus V is finite-dimensional. The action of F^r on V is injective and therefore bijective (by Proposition 3.4.1).

The converse is immediate. □

Proposition 3.4.10 *Let L be a (not necessarily perfect) field that contains k. Let V be a unit $L[F^r]$-module. Then V is finite-unit if and only if it is finite-dimensional over L.*

Proof. This follows by the same reasoning as in the proof of Proposition 3.4.8. If V is finite-unit then it has a root, and this root must be V itself. □

Proposition 3.4.11 *Let L be a separably closed field containing k. Let V be a finite-unit $L[F^r]$-module. Let Q be the $\mathbb{F}_q$-subspace of V consisting of elements fixed by F^r. Then the homomorphism*

$$Q \otimes_{\mathbb{F}_q} L \to V \tag{3.4.12}$$

is an isomorphism.

Proof. This is Proposition 1.1 of Exposé XXII in [3]. □

Proposition 3.4.13 *Let L be an algebraically closed field containing k. Let V be a left $L[F^r]$-module which is finite-dimensional over L. Let Q be the $\mathbb{F}_q$-subspace of V consisting of elements fixed by F^r. Then the homomorphism*

$$\left(Q \otimes_{\mathbb{F}_q} L\right) \oplus V^{nil} \to V \tag{3.4.14}$$

is an isomorphism.

Proof. This is a consequence of Proposition 3.4.3 and Proposition 3.4.11. □

Proposition 3.4.15 *Let L be an algebraically closed field containing k. Let V be a left $L[F^r]$-module that is finite-dimensional over L. Let Q be the $\mathbb{F}_q$-subspace of V consisting of elements fixed by F^r. Then $\dim_{\mathbb{F}_q} Q = \dim_L V^{surj}$, and the sequence*

$$0 \longrightarrow Q \longrightarrow V \xrightarrow{1-F^r} V \longrightarrow 0 \tag{3.4.16}$$

is exact.

Proof. The only assertion that requires proof is the assertion that the action of $1 - F^r$ on V is surjective. By Proposition 3.4.13, it suffices to prove this assertion in the case where $V \cong L^{\oplus d}$ for some $d \geq 0$, and in the case where $V = V^{nil}$. In the first case, the fact that L is algebraically closed implies that the map $Id_L - F_L^r \colon L \to L$ is surjective. In the second case, the action of $(1 + F^r + \dots F^{rn})$ on V is inverse to the action of $1 - F^r$ for sufficiently large n. □

Chapter 4

Local analysis of $\mathcal{O}_{F^r,X}$-modules on a curve

Let $\mathbb{A}^1 = \text{Spec } k[t]$, and let A be the stalk of the (étale) structure sheaf $\mathcal{O}_{\mathbb{A}^1}$ at (t). Let K denote the fraction field of A. As in Chapter 3, let r be a positive integer and let $q = p^r$. The objects of study in this chapter are finite-unit $A[F^r]$-modules.

We summarize here some of the properties of A and K. (See Sections I.1 and I.2 of [2] for a more extensive discussion.) The ring A is a Henselization of the local ring $k[t]_{(t)}$. It is isomorphic to the ring of elements of the power series ring $k[[t]]$ that are algebraic over $k(t)$. The field K is a maximal unramified extension of $k(t)$ for the ideal $(t) \subseteq k[t]$. Thus, any finite extension of K is totally ramified at $(t) \subseteq A$. If K_0/K is a finite extension and $A_0 \subseteq K_0$ is the integral closure of A inside K_0, then the local ring A_0 is also Henselian (see Lemma 3.1 in [2]). Moreover, there exists an isomorphism between A_0 and A. (If t_0 is a local parameter for A_0, then the k-homomorphism $k(t) \hookrightarrow K_0$ which sends t to t_0 extends to an isomorphism $K \to K_0$ which maps A isomorphically onto A_0.)

Let K_0/K be a field extension of degree d, let $A_0 \subseteq K_0$ be the integral closure of A, and let $t_0 \in A_0$ be a local parameter for A_0. Then the inclusion $A \hookrightarrow A_0$ makes A_0 a free A-module of rank d with basis $\left\{1, t_0, t_0^2, \ldots, t_0^{d-1}\right\}$. In particular, since the integral closure of the image of the A under the Frobenius homomorphism

$$F_K^r \colon K \to K \tag{4.0.1}$$

is A itself, the Frobenius homomorphism $F_A^r \colon A \to A$ makes A a free A-module with basis $\{1, t, t^2, \ldots, t^{q-1}\}$.

For discussions of finite-unit $A[F^r]$-modules it is helpful to introduce a second copy of the ring A. Let A' denote a copy of A with a right A-module structure given by $F_A^r \colon A \to A$ and a left A-module structure given by the identity map. Then if W is an A-module, a left $A[F^r]$-module structure on W is uniquely determined by an A-module homomorphism

$$A' \otimes_A W \to W. \tag{4.0.2}$$

We refer to this homomorphism as the structural homomorphism for W.

The following lemma will be useful for proofs in this section.

Lemma 4.0.3 *Let W be a finite-unit $A[F^r]$-module. For each $w \in W$, there exists a unique q-tuple $(w_0, \ldots, w_{q-1}) \in W^{\oplus q}$ such that*

$$w = \sum_{i=0}^{q-1} t^i F^r(w_i). \tag{4.0.4}$$

Proof. Since the homomorphism $F_A^r\colon A \to A$ makes A a free A-module with basis $\{1, t, t^2, \ldots, t^{q-1}\}$, each element of $A' \otimes_A W$ may be uniquely expressed as a sum

$$\sum_{i=0}^{q-1} t^i \otimes w_i, \tag{4.0.5}$$

with $w_i \in W$. The lemma follows from the fact that the structural homomorphism

$$A' \otimes_A W \to W \tag{4.0.6}$$

is an isomorphism. □

If W is a left $A[F^r]$-module, the kernel of the homomorphism

$$W \to K \otimes_A W \tag{4.0.7}$$

is the A-torsion submodule of W. This chapter is primarily concerned with left $A[F^r]$-modules that are torsion-free. If W is a torsion-free left $A[F^r]$-module, we will treat W as a submodule of $K \otimes_A W$.

4.1 The structure of a finite-unit $A[F^r]$-module

Proposition 4.1.1 *Let W be a left $A[F^r]$-module that is free of finite rank as an A-module. Let $V \subseteq W$ be the $\mathbb{F}_q$-vector space consisting of elements that are fixed by F^r. The sequences*

$$0 \longrightarrow V \longrightarrow W \xrightarrow{1-F^r} W \longrightarrow 0 \tag{4.1.2}$$

and

$$0 \longrightarrow V \longrightarrow W/tW \xrightarrow{1-F^r} W/tW \longrightarrow 0 \tag{4.1.3}$$

are exact.

Proof. Consider the diagram

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & tW & \xrightarrow{1-F^r} & tW \\ \downarrow & & \downarrow & & \downarrow \\ V & \longrightarrow & W & \xrightarrow{1-F^r} & W \\ \downarrow{\scriptstyle Id} & & \downarrow & & \downarrow \\ V & \longrightarrow & W/tW & \xrightarrow{1-F^r} & W/tW \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0. \end{array} \tag{4.1.4}$$

Lemma 4.1.5 *The map*

$$tW \xrightarrow{1-F^r} tW \tag{4.1.6}$$

is bijective.

Proof. Fix a $k[t]$-algebra homomorphism $A \hookrightarrow k[[t]]$ which maps A onto the subring of elements of $k[[t]]$ that are algebraic over $k(t)$. Note that $F^r(tW) \subseteq t^q W$. There is a well-defined map

$$\sum_{n=0}^{\infty} F^{rn} \colon k[[t]] \otimes_A (tW) \to k[[t]] \otimes_A (tW) \tag{4.1.7}$$

which is an inverse to the action of $1 - F^r$. To prove the lemma it suffices to show that if a is an element of $A \otimes_A (tW)$, then the element

$$b = \sum_{n=0}^{\infty} F^{rn} a \in k[[t]] \otimes_A (tW) \tag{4.1.8}$$

is contained in $A \otimes_A (tW)$.

Let $\overline{k((t))}$ be an algebraic closure of the fraction field of $k[[t]]$. Let $\overline{K}$ be the algebraic closure of K inside of $\overline{k((t))}$. Proposition 3.4.15 applied to $\overline{k((t))} \otimes_A (tW)$ and to

$\overline{K} \otimes_A (tW)$ shows that the sizes of the fibers of the map

$$(1 - F^r)\colon \overline{k((t))} \otimes_A (tW) \to \overline{k((t))} \otimes_A (tW) \tag{4.1.9}$$

and the sizes of the fibers of the restriction

$$(1 - F^r)\colon \overline{K} \otimes_A (tW) \to \overline{K} \otimes_A (tW) \tag{4.1.10}$$

are all the same. It follows that the pre-image of $\overline{K} \otimes_A (tW)$ under map (4.1.9) is exactly $\overline{K} \otimes_A (tW)$. In the previous paragraph,

$$(1 - F^r)b = a \in \overline{K} \otimes_A (tW), \tag{4.1.11}$$

so $b \in \overline{K} \otimes_A (tW)$ and therefore $b \in A \otimes_A (tW)$. □

Lemma 4.1.12 *The map*

$$V \to W/tW \tag{4.1.13}$$

is injective.

Proof. Let $v \in V \cap tW$. Since v is fixed by F^r, for any $n \geq 0$,

$$v = F^{rn} v \in t^{q^n} W. \tag{4.1.14}$$

Therefore $v = 0$. The lemma is proved. □

Lemma 4.1.15 *The map*

$$W/tW \xrightarrow{1-F^r} W/tW \tag{4.1.16}$$

is surjective.

Proof. The quotient W/tW is a left $k[F^r]$-module that is finite-dimensional over k. The action of $1 - F^r$ on W/tW is surjective by Proposition 3.4.15. □

The Snake Lemma implies that in diagram (4.1.4) the map

$$W \xrightarrow{1-F^r} W \tag{4.1.17}$$

is surjective, and that the kernel of

$$W/tW \xrightarrow{1-F^r} W/tW \tag{4.1.18}$$

is the image of $V \to W/tW$. This completes the proof of Proposition 4.1.1. □

Definition 4.1.19 *Let W be a torsion-free unit $A[F^r]$-module. Then $W^{vec} \subseteq W$ is the subset consisting of elements w such that $t^{-n}w \in K \otimes_A W$ is contained in W for all $n > 0$.*

If W is a torsion-free unit $A[F^r]$-module, W^{vec} is the largest K-vector space contained in W. It is easily seen that W^{vec} is stabilized by the action of F^r. The K-vector space W^{vec} has the structure of a unit $K[F^r]$-module.

If W is a torsion-free unit $A[F^r]$-module and $A \hookrightarrow A_0$ is a finite integral extension of A, then $A_0 \otimes_A W$ is a torsion-free unit $A_0[F^r]$-module. Let K_0 be the fraction field of A_0. We similarly denote by $(A_0 \otimes_A W)^{vec}$ the largest K_0-vector space contained in $A_0 \otimes_A W$.

Proposition 4.1.20 *Let W be a torsion-free finite-unit $A[F^r]$-module, and let $A \hookrightarrow A_0$ be a finite integral extension. Then*

$$(A_0 \otimes_A W)^{vec} = (A_0 \otimes_A W^{vec}) \tag{4.1.21}$$

in $A_0 \otimes_A W$.

Proof. Let t_0 be a local parameter for the discrete valuation ring A_0. Let K_0 be the fraction field of A_0. Let g be the degree of the extension K_0/K. Any element $w \in A_0 \otimes_A W$ may be *uniquely* expressed as a sum

$$w = \sum_{i=0}^{g-1} t_0^i \otimes w_i \tag{4.1.22}$$

with $w_i \in W$.

Let v be an element of $(A_0 \otimes_A W)^{vec}$. Let $v_0, \ldots, v_{g-1} \in W$ be the elements such that

$$v = \sum_{i=0}^{g-1} t_0^i \otimes v_i. \tag{4.1.23}$$

The set $(A_0 \otimes_A W)^{vec}$ has the structure of a K_0-vector space. Thus if n is any nonnegative integer, then there exists $v' \in (A_0 \otimes_A W)^{vec}$ such that $t^n v' = v$. (Recall that t denotes the local parameter for A.) If $(v'_0, \ldots, v'_{g-1})$ is the unique g-tuple of elements of W such that

$$v' = \sum_{i=0}^{g-1} t_0^i \otimes v'_i, \tag{4.1.24}$$

then $t^n v'_i = v_i$. We conclude that each v_i is contained in W^{vec}. Therefore $v \in A_0 \otimes_A W^{vec}$.

We have shown that

$$(A_0 \otimes_A W)^{vec} \subseteq (A_0 \otimes_A W^{vec}). \tag{4.1.25}$$

The reverse inclusion is obvious, since $A_0 \otimes_A W^{vec}$ is a K_0-vector space contained in $A_0 \otimes_A W$. □

Note that for any $e \geq 0$, the A-modules $A^{\oplus e}$ and $K^{\oplus e}$ have finite-unit $A[F^r]$-module structures (given by the qth power maps on A and K, respectively).

Proposition 4.1.26 *Let W be a finite-unit $A[F^r]$-module such that there exists a left $K[F^r]$-module isomorphism*

$$K \otimes_A W \cong K^{\oplus d} \tag{4.1.27}$$

for some $d \geq 0$. Then for some d' with $0 \leq d' \leq d$, there exists a left $A[F^r]$-module isomorphism

$$W \cong K^{\oplus d'} \oplus A^{\oplus (d-d')}. \tag{4.1.28}$$

Proof. Let $\{e_1, \ldots, e_d\} \subseteq K \otimes_A W$ be the basis which determines isomorphism (4.1.27). We show that this basis is contained in W. Suppose, for the sake of contradiction, that $e_i \notin W$ for some i with $1 \leq i \leq d$. Let n be the smallest positive integer such that $t^n e_i \in W$. Let

$$n = \ell q + s, \tag{4.1.29}$$

where ℓ is an integer and s is a nonnegative integer less than q. Then

$$t^n e_i = t^s F^r(t^\ell e_i), \tag{4.1.30}$$

and $\ell < n$. By Lemma 4.0.3 applied to W, there exists a unique expression

$$t^n e_i = \sum_{i=0}^{q-1} t^i F^r(w_i) \tag{4.1.31}$$

with $w_i \in W$. By Lemma 4.0.3 applied to $K \otimes_A W$, formula (4.1.31) coincides with formula (4.1.30). Therefore $t^\ell e_i = w_i \in W$, which yields a contradiction.

We have shown that $\{e_1, \ldots, e_d\} \subseteq W$. Let $V \subseteq W$ be the $\mathbb{F}_q$-subspace spanned by $\{e_1, \ldots, e_d\}$, and let $V' = V \cap tW$. Choose an $\mathbb{F}_q$-basis $\{e'_1, \ldots, e'_{d'}\}$ for V' (where

$d' = \dim_k V'$), and choose elements $e'_{d'+1}, \ldots, e'_d \in V \setminus V'$ so that $\{e'_1, \ldots, e'_d\}$ is an $\mathbb{F}_q$-basis for V. For any i with $1 \leq i \leq d'$, the element $t^{-1}e'_i \in K \otimes_A W$ is contained in W, and therefore $F^{rn}(t^{-1}e'_i) = t^{-q^n}e'_i$ is contained in W for any $n > 0$. For any i with $d' < i \leq d$, the element $e'_i \in K \otimes_A W$ is contained in W but the element $t^{-1}e'_i \in K \otimes_A W$ is not contained in W. The basis $\{e'_1, \ldots, e'_d\}$ for W determines isomorphism (4.1.28). □

Corollary 4.1.32 *Let $A \hookrightarrow A_0$ be a finite integral extension, and let K_0/K be the corresponding extension of fraction fields. Let W be a finite-unit $A[F^r]$-module such that there exists a left $K_0[F^r]$-module isomorphism*

$$K_0 \otimes_A W \cong K_0^{\oplus d} \tag{4.1.33}$$

for some $d \geq 0$. Then for some d' with $0 \leq d' \leq d$, there exists a left $A_0[F^r]$-module isomorphism

$$A_0 \otimes_A W \cong K_0^{\oplus d'} \oplus A_0^{\oplus(d-d')}. \tag{4.1.34}$$

Proof. This corollary follows immediately via the fact that A and A_0 are isomorphic as k-algebras. □

Proposition 4.1.35 *Let W be a torsion-free finite-unit $A[F^r]$-module. If $W^{vec} = \{0\}$, then W is isomorphic as a left $A[F^r]$-module to $A^{\oplus d}$ for some $d \geq 0$.*

Proof. We first show that W is finitely-generated as an A-module. Let K^{sep} be a separable closure of K. By Proposition 3.4.11, the K^{sep}-vector space $K^{sep} \otimes_A W$ contains a basis that is fixed by F^r. Choose a finite subextension K_0/K of K^{sep}/K such that $K_0 \otimes_A W$ contains this basis. Let A_0 be the integral closure of A inside of K_0. There exists an isomorphism of left $K_0[F^r]$-modules

$$K_0 \otimes_A W \cong K_0^{\oplus d} \tag{4.1.36}$$

for some $d \geq 0$. By Corollary 4.1.32, there exists an isomorphism of left $A_0[F^r]$-modules

$$A_0 \otimes_A W \cong K_0^{\oplus d'} \oplus A_0^{\oplus (d-d')} \tag{4.1.37}$$

for some d' with $0 \leq d' \leq d$. However Proposition 4.1.20 implies that the K_0-dimension of $(A_0 \otimes_A W)^{vec}$ is equal to the K-dimension of W^{vec}, which is zero. Therefore $d' = 0$ in expression (4.1.37), and in fact

$$A_0 \otimes_A W \cong A_0^{\oplus d}. \tag{4.1.38}$$

Thus $A_0 \otimes_A W$ is a finitely-generated A_0-module. Since A_0 is finite over A, it follows that $A_0 \otimes_A W$ is also finitely-generated as an A-module. The sub-A-module $W \subseteq A_0 \otimes_A W$ is therefore likewise finitely-generated.

Let Q be the $\mathbb{F}_q$-subspace of W that is fixed by F^r. The sequence

$$0 \longrightarrow Q \longrightarrow W/tW \xrightarrow{1-F^r} W/tW \longrightarrow 0 \tag{4.1.39}$$

is exact by Proposition 4.1.1. Since W is a finite-unit $A[F^r]$-module, W/tW is a finite-unit $k[F^r]$-module. The image of Q spans W/tW as a k-vector space by Proposition 3.4.11. Let $\{q_1, \dots, q_d\} \subseteq Q$ be an $\mathbb{F}_q$-basis for Q. The image of $\{q_1, \dots, q_d\}$ in (W/tW) is a k-basis for (W/tW), and therefore $\{q_1, \dots, q_d\}$ is itself an A-basis for W. The A-basis $\{q_1, \dots, q_d\}$ determines a left $A[F^r]$-module isomorphism $W \cong A^{\oplus d}$. □

Corollary 4.1.40 *Let W be a torsion-free unit $A[F^r]$-module. Then W is finite-unit if and only if W^{vec} is a finite-dimensional K-vector space and $W/W^{vec} \cong A^{\oplus d}$ for some $d \geq 0$.*

Proof. This follows easily from Propositions 4.1.35 and 3.4.10. □

4.2 The F^r-index

Let W be a torsion-free finite-unit $A[F^r]$-module. We say that an A-submodule $W_0 \subseteq W$ *spans* W if $K \otimes_A W_0 = K \otimes_A W$. This section defines a numerical invariant associated to finitely-generated A-submodules that span W.

Lemma 4.2.1 *Let W be a torsion-free finite-unit $A[F^r]$-module. Let $W_0, W_0' \subseteq W$ be finitely-generated A-submodules that span W such that $W_0' \subseteq W_0$. Then W_0/W_0' is finite-dimensional as a k-vector space.*

Proof. Since W_0' spans W, there exists n such that $t^n W_0 \subseteq W_0'$. The k-dimension of W_0/W_0' is bounded by $n \cdot \dim_K (K \otimes_A W)$. □

Definition 4.2.2 *Let W be a torsion-free finite-unit $A[F^r]$-module. Let $W_0 \subseteq W$ be a finitely-generated A-submodule that spans W. Let $W_1 \subseteq W$ be the A-submodule of W generated by $F^r(W_0)$. Then the F^r-*index *of W_0 inside W is*

$$\mathrm{ind}_W W_0 = \frac{\dim_k (W_1/W_1 \cap W_0) - \dim_k (W_0/W_1 \cap W_0)}{q-1}. \tag{4.2.3}$$

(When the containing module W is clear from the context we may omit the subscript and simply write ind W_0*.)*

Note that $\mathrm{ind}_W W_0$ is always a rational whose denominator divides $q-1$.

Proposition 4.2.4 *Let W be a torsion-free finite-unit $A[F^r]$-module. Let $W_0, W_0' \subseteq W$ be finitely-generated A-submodules that span W such that $W_0' \subseteq W_0$. Then*

$$\dim_k(W_0/W_0') = \mathrm{ind}\ W_0 - \mathrm{ind}\ W_0'. \tag{4.2.5}$$

Proof. Let $W_1, W_1' \subseteq W$ be the A-submodules generated by $F^r(W_0)$ and $F^r(W_0')$, respectively. Note that in the definition of ind W_0, the module $W_1 \cap W_0$ could be replaced by any spanning A-submodule of W that is contained in both W_1 and W_0. For, if $N \subseteq W_1 \cap W_0$ spans W, then

$$\begin{aligned}
\text{ind } W_0 &= \frac{\dim_k (W_1/W_1 \cap W_0) - \dim_k (W_0/W_1 \cap W_0)}{q-1} \\
&= \frac{\dim_k (W_1/N) - \dim_k(W_0 \cap W_1/N) - \dim_k (W_0/N) + \dim_k(W_0 \cap W_1/N)}{q-1} \\
&= \frac{\dim_k (W_1/N) - \dim_k (W_0/N)}{q-1}.
\end{aligned}$$

Apply this with $N = W_0 \cap W_1 \cap W_0' \cap W_1'$. Then

$$\begin{aligned}
\text{ind } W_0 - \text{ind } W_0' &= \frac{\dim_k(W_1/N) - \dim_k(W_0/N) - \dim_k(W_1'/N) + \dim_k(W_0'/N)}{q-1} \\
&= \frac{\dim_k(W_1/N) - \dim_k(W_1'/N) - \dim_k(W_0/N) + \dim_k(W_0'/N)}{q-1} \\
&= \frac{\dim_k(W_1/W_1') - \dim_k(W_0/W_0')}{q-1}.
\end{aligned}$$

The A-modules W_1 and W_1' are, respectively, the images under the structural homomorphism

$$A' \otimes_A W \to W \tag{4.2.6}$$

of $A' \otimes_A W_0$ and $A' \otimes_A W_0'$. (See the beginning of this chapter for the definition of A'.) Since (4.2.6) is an isomorphism, the homomorphisms

$$A' \otimes_A W_0 \to W_1 \text{ and } A' \otimes_A W_0' \to W_1' \tag{4.2.7}$$

are isomorphisms. By the 5-lemma, the homomorphism

$$\left(A' \otimes_A W_0\right) / \left(A' \otimes_A W_0'\right) \to W_1/W_1'. \tag{4.2.8}$$

is likewise an isomorphism. Since A' is a free right A-module of rank q, the k-dimension of

$$\left(A' \otimes_A W_0\right) / \left(A' \otimes_A W_0'\right) \cong A' \otimes_A \left(W_0/W_0'\right) \tag{4.2.9}$$

is q times the k-dimension of W_0/W_0'. Therefore,

$$\text{ind } W_0 - \text{ind } W_0' = \frac{(q-1)\dim_k\left(W_0/W_0'\right)}{q-1} = \dim_k\left(W_0/W_0'\right), \tag{4.2.10}$$

as desired. □

Proposition 4.2.11 *Let W be a torsion-free finite-unit $A[F^r]$-module. Let $W_0 \subseteq W$ be a finitely-generated A-submodule that spans W. Let $W_1 \subseteq W$ be the A-submodule generated by $F^r(W_0)$. Then,*

$$\text{ind } W_1 = q \cdot \text{ind } W_0. \tag{4.2.12}$$

Proof. Let $W_2 \subseteq W$ be the A-submodule generated by $F^r(W_1)$. Note that because A' is a free right A-module,

$$(A' \otimes_A W_0) \cap (A' \otimes_A W_1) = A' \otimes_A (W_0 \cap W_1) \tag{4.2.13}$$

in $A' \otimes_A W$. Therefore,

$$\begin{aligned}
\text{ind } W_1 &= \frac{\dim_k(W_2/W_1 \cap W_2) - \dim_k(W_1/W_1 \cap W_2)}{q-1} && (4.2.14)\\
&= \frac{\dim_k((A' \otimes_A W_1)/(A' \otimes_A W_0) \cap (A' \otimes_A W_1))}{q-1} && (4.2.15)\\
&\quad - \frac{\dim_k((A' \otimes_A W_0)/(A' \otimes_A W_0) \cap (A' \otimes_A W_1))}{q-1} && (4.2.16)\\
&= \frac{\dim_k A' \otimes_A (W_1/W_1 \cap W_0) - \dim_k A' \otimes_A (W_0/W_1 \cap W_0)}{q-1} && (4.2.17)\\
&= \frac{q \dim_k W_1/W_1 \cap W_0 - q \dim_k W_0/W_1 \cap W_0}{q-1} && (4.2.18)\\
&= q \cdot \text{ind } W_0, && (4.2.19)
\end{aligned}$$

as desired. □

4.3 The minimal root of a finite-unit $A[F^r]$-module

Let W be a torsion-free finite-unit $A[F^r]$-module. Recall (Definition 3.3.1) that an A-submodule $W_0 \subseteq W$ is a *root* if

1. W_0 is finitely generated as an A-module,
2. the A-submodule of W generated by $F^r(W_0)$ contains W_0, and
3. W is generated as a left $A[F^r]$-module by W_0.

Theorem 4.3.1 *Let W be a torsion-free finite-unit $A[F^r]$-module. Then W has a root which is contained in all other roots.*

A root that is contained in all other roots is called a "minimal root." Theorem 4.3.1 is a consequence of several small lemmas.

Lemma 4.3.2 *There exists a root for W^{vec}.*

Proof. Let $\{w_1, \ldots, w_e\} \subseteq W^{vec}$ be a K-basis for W^{vec}. Let W_0 and W_1 be the A-submodules of W generated by

$$\{w_1, \ldots, w_e\} \text{ and } \{F^r w_1, \ldots, F^r w_e\}, \tag{4.3.3}$$

respectively. Since W^{vec} is a unit $K[F^r]$-module, both W_0 and W_1 span W^{vec}. Choose n large enough that

$$t^{n(q-1)-1} W_0 \subseteq W_1. \tag{4.3.4}$$

Let $W_0' = t^{-n} W_0$. The A-submodule of W^{vec} generated by $F^r(W_0')$ contains $t^{-1} W_0'$. This fact implies that W_0' is a root for W^{vec}. □

Lemma 4.3.5 *There exists a root for W.*

Proof. By Corollary 4.1.40, there exists a left $A[F^r]$-module isomorphism

$$W/W^{vec} \cong A^{\oplus d} \tag{4.3.6}$$

for some $d \geq 0$. Let $\{w_1, \ldots, w_d\} \subseteq W/W^{vec}$ be the basis which determines this isomorphism, and choose elements $w_1', \ldots, w_d' \in W$ which map to $w_1, \ldots, w_d$. Then $F^r(w_i') - w_i' \in W^{vec}$ for any $i \in \{1, \ldots, d\}$.

Let $W_0 \subseteq W^{vec}$ be a root for W^{vec}. Let n be the smallest positive integer such that $t^n e_i \in W$.

$$F^r(w_i') - w_i' \in t^{-n} W_0 \tag{4.3.7}$$

for every i. Let $W_0' \subseteq W$ be the A-submodule generated by $t^{-n}W_0$ and $\{w_1', \ldots, w_n'\}$. Then W_0' is a root for W. □

Lemma 4.3.8 *The intersection of two roots in W is a root.*

Proof. Let $W_0, W_0' \subseteq W$ be roots. Let

$$W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \tag{4.3.9}$$

$$W_0' \subseteq W_1' \subseteq W_2' \subseteq \ldots \tag{4.3.10}$$

be the filtrations for W described in Proposition 3.3.4.

Note that $W_1 \cap W_1'$ is the image of

$$(A' \otimes_A W_0) \cap (A' \otimes_A W_0') = A' \otimes_A (W_0 \cap W_0') \tag{4.3.11}$$

under the structural morphism for W. The intersection of W_1 and W_1' is the A-submodule of W generated by $F^r(W_0 \cap W_0')$. Similarly, for each $n > 1$, $W_n \cap W_n'$ is the A-submodule of W generated by $F^{rn}(W_0 \cap W_0')$. Since the sequences $(W_n)_{n=0}^{\infty}$ and $(W_n')_{n=0}^{\infty}$ are ascending filtrations for W, the sequence $(W_n \cap W_n')_{n=0}^{\infty}$ is an ascending filtration for W. Therefore $W_0 \cap W_0'$ is a root of W. □

Lemma 4.3.12 *If $W_0 \subseteq W$ is a root, then W_0 spans W.*

Proof. Since W_0 is a root for W, $K \otimes_A W_0$ is a root for $K \otimes_A W$. But a root for a finite-unit $K[F^r]$-module can only be the $K[F^r]$-module itself. □

Lemma 4.3.13 *If $W_0 \subseteq W$ is a root, then* $\operatorname{ind} W_0 \geq 0$.

Proof. Let W_1 be the A-submodule of W generated by $F^r(W_0)$. Then

$$\operatorname{ind} W_0 = \frac{\dim_k (W_1/W_1 \cap W_0)}{q-1} - \frac{\dim_k (W_0/W_1 \cap W_0)}{q-1} \tag{4.3.14}$$

(Definition 4.2.2). Since $W_0 \subseteq W_1$, the second term is zero. □

Proof of Theorem 4.3.1. The F^r-index of any root of W is a nonnegative rational with denominator less than or equal to $q-1$. Let W_0 be a root in W with minimal F^r-index. If W_0' is another root of W, then $W_0' \cap W_0$ is a root of W whose F^r-index cannot be less than that of W_0. By Proposition 4.2.4, $W_0' \cap W_0 = W_0$. Therefore $W_0 \subseteq W_0'$. □

Definition 4.3.15 *Let W be a torsion-free finite-unit $A[F^r]$-module. The* minimal root index *of W is the F^r-index of the smallest root in W.*

Proposition 4.3.16 *Let W be a finite-unit $A[F^r]$-module which is free of rank d as an A-module. Then the only root of W is W itself. The minimal root index of W is 0.*

Proof. By Proposition 4.1.35, W is isomorphic as a left $A[F^r]$-module to $A^{\oplus d}$. If W_0 is a root for W, then the image of

$$W_0 \to W/tW \tag{4.3.17}$$

is a root for W/tW. The above map must be surjective and therefore by Nakayama's lemma, $W_0 = W$. □

4.4 The local Riemann-Hilbert correspondence

As in previous sections, A denotes the stalk of the structure sheaf of Spec $k[t]$ at (t), and K denotes the fraction field of A. This section is concerned with a correspondence between finite-unit $A[F^r]$-modules and constructible étale sheaves on Spec A. It is convenient at this point to revert to the language of $\mathcal{O}_{F^r,X}$-modules. Let $Z = \text{Spec } A$. Let s be the closed point of Z and let η be the generic point of Z. Let K^{sep} be a separable closure of K, and let

$$\overline{\eta}\colon \text{Spec } K^{sep} \to Z \tag{4.4.1}$$

be the corresponding geometric point.

Note that the structure sheaf $\mathcal{O}_Z$ has a natural left $\mathcal{O}_{F^r,Z}$-module structure. If f is a section of $\mathcal{O}_Z$, then

$$F^r(f) = f^q. \tag{4.4.2}$$

If $\mathcal{M}$ is another left $\mathcal{O}_{F^r,Z}$-module, we write

$$\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{M}, \mathcal{O}_Z) \tag{4.4.3}$$

for the sheaf of left $\mathcal{O}_{F^r,Z}$-module morphisms from $\mathcal{M}$ to $\mathcal{O}_Z$. This is a sheaf of $\mathbb{F}_q$-vector spaces on Z. Conversely, if M is an $\mathbb{F}_q$-étale sheaf on Z, then the sheaf

$$\mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_Z) \tag{4.4.4}$$

has a left $\mathcal{O}_{F^r,Z}$-module structure determined by the left $\mathcal{O}_{F^r,Z}$-module structure of $\mathcal{O}_Z$.

We begin by establishing a correspondence over $\{\eta\}$.

Theorem 4.4.5 *Let $\mathcal{V}$ be a finite-unit $\mathcal{O}_{F^r,\{\eta\}}$-module. Then*

$$\mathcal{H}om_{\mathcal{O}_{F^r,\{\eta\}}}\left(\mathcal{V}, \mathcal{O}_{\{\eta\}}\right) \tag{4.4.6}$$

is an $\mathbb{F}_q$-étale sheaf whose stalk is of the same dimension as the stalk of $\mathcal{V}$. The double-dual homomorphism

$$\mathcal{V} \to \mathcal{H}om_{\mathbb{F}_q}\left(\mathcal{H}om_{\mathcal{O}_{F^r,\{\eta\}}}\left(\mathcal{V}, \mathcal{O}_{\{\eta\}}\right), \mathcal{O}_{\{\eta\}}\right) \tag{4.4.7}$$

is an isomorphism.

Let V be a constructible $\mathbb{F}_q$-étale sheaf on $\{\eta\}$. Then

$$\mathcal{H}om_{\mathbb{F}_q}\left(V, \mathcal{O}_{\{\eta\}}\right) \tag{4.4.8}$$

is a finite-unit $\mathcal{O}_{F^r,\{\eta\}}$-module whose stalk has the same dimension as the stalk of V. The double-dual homomorphism

$$V \to \mathcal{H}om_{\mathcal{O}_{F^r,\{\eta\}}}\left(\mathcal{H}om_{\mathbb{F}_q}\left(V, \mathcal{O}_{\{\eta\}}\right), \mathcal{O}_{\{\eta\}}\right) \tag{4.4.9}$$

is an isomorphism.

Proof. By Proposition 3.4.11, there exists a left $K^{sep}[F^r]$-module isomorphism

$$\mathcal{V}_{\overline{\eta}} \cong (K^{sep})^{\oplus e} \tag{4.4.10}$$

for some $e \geq 0$. It is easily seen that the stalks

$$\mathrm{Hom}_{K^{sep}[F^r]}\left(\mathcal{V}_{\overline{\eta}}, K^{sep}\right), \tag{4.4.11}$$

$$\mathrm{Hom}_{\mathbb{F}_q}\left(V_{\overline{\eta}}, K^{sep}\right) \tag{4.4.12}$$

of (4.4.6) and (4.4.8) have the appropriate dimensions, and that the stalk homomorphisms

$$\mathcal{V}_{\overline{\eta}} \to \mathrm{Hom}_{\mathbb{F}_q}\left(\mathrm{Hom}_{K^{sep}[F^r]}\left(\mathcal{V}_{\overline{\eta}}, K^{sep}\right), K^{sep}\right) \tag{4.4.13}$$

$$V_{\overline{\eta}} \to \mathrm{Hom}_{K^{sep}[F^r]}\left(\mathrm{Hom}_{\mathbb{F}_q}\left(V_{\overline{\eta}}, K^{sep}\right), K^{sep}\right) \tag{4.4.14}$$

of (4.4.7) and (4.4.9) are isomorphisms. □

If $\mathcal{N}$ is a torsion-free finite-unit $\mathcal{O}_{F^r,Z}$-module, let $\mathcal{N}^{vec}$ denote the subsheaf generated by the K-vector space $\Gamma(Z,\mathcal{N})^{vec} \subseteq \Gamma(Z,\mathcal{N})$ (recall Definition 4.1.19). By Corollary 4.1.40, $\mathcal{N}/\mathcal{N}^{vec}$ is isomorphic as a left $\mathcal{O}_{F^r,Z}$-module to $\mathcal{O}_Z^{\oplus d}$ for some $d \geq 0$.

Theorem 4.4.15 *Let $\mathcal{M}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,Z}$-module. Then*

$$M' = \mathcal{H}om_{\mathcal{O}_{F^r,Z}}\left(\mathcal{M}, \mathcal{O}_Z\right) \tag{4.4.16}$$

is a constructible $\mathbb{F}_q$-étale sheaf on Z whose sections all have open support. The $\mathbb{F}_q$-dimension of $\Gamma(Z, M')$ is equal to the $\mathcal{O}_Z$-rank of $\mathcal{M}/\mathcal{M}^{vec}$. The $\mathbb{F}_q$-dimension of $M'_{\overline{\eta}}$ is equal to the K-rank of $\mathcal{M}_\eta$. The double-dual homomorphism

$$\mathcal{M} \to \mathcal{H}om_{\mathbb{F}_q}\left(M', \mathcal{O}_Z\right) \tag{4.4.17}$$

is an isomorphism.

Let M be a constructible $\mathbb{F}_q$-étale sheaf on Z whose sections all have open support. Then

$$\mathcal{M}' = \mathcal{H}om_{\mathbb{F}_q}\left(M, \mathcal{O}_Z\right) \tag{4.4.18}$$

is a torsion-free finite-unit $\mathcal{O}_{F^r,Z}$-module. The $\mathcal{O}_Z$-rank of $\mathcal{M}'/\mathcal{M}'^{vec}$ is equal to the $\mathbb{F}_q$-dimension of $\Gamma(Z, M)$. The K-rank of $\mathcal{M}_\eta$ is equal to the $\mathbb{F}_q$-dimension of $M_{\overline{\eta}}$. The double-dual homomorphism

$$M \to \mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{M}', \mathcal{O}_Z) \tag{4.4.19}$$

is an isomorphism.

The proof of Theorem 4.4.15 uses the following two lemmas. If N is a constructible $\mathbb{F}_q$-étale sheaf, let $N^{con} \subseteq N$ denote the subsheaf generated by global sections.

Lemma 4.4.20 *Let N be a constructible $\mathbb{F}_q$-étale sheaf whose sections all have open support. Then*

$$\mathcal{H}om_{\mathbb{F}_q}(N, \mathcal{O}_Z)^{vec} \tag{4.4.21}$$

is the subsheaf of $\mathcal{H}om_{\mathbb{F}_q}(N, \mathcal{O}_Z)$ consisting of morphisms that kill $N^{con} \subseteq N$.

Proof. A morphism $\phi\colon N \to \mathcal{O}_Z$ may be expressed as a diagram

$$\begin{array}{ccc} N_{\overline{\eta}} & \longrightarrow & K^{sep} \\ \uparrow & & \uparrow \\ N_s & \longrightarrow & A \end{array} \tag{4.4.22}$$

in which the the top map is $\mathrm{Gal}(K^{sep}/K)$-equivariant. A morphism is contained in a K-vector space of morphisms $N \to \mathcal{O}_Z$ if and only if the bottom map is zero. □

Lemma 4.4.23 *Let $\mathcal{N}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,Z}$-module. Then*

$$\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{N}, \mathcal{O}_Z)^{con} \tag{4.4.24}$$

is the subsheaf of $\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{N}, \mathcal{O}_Z)$ consisting of morphisms that kill $\mathcal{N}^{vec} \subseteq \mathcal{N}$.

Proof. Since $\mathcal{N}/\mathcal{N}^{vec} \cong \mathcal{O}_Z^{\oplus d}$ for some $d \geq 0$, the sheaf of left $\mathcal{O}_{F^r,Z}$-module morphisms from $\mathcal{N}$ to $\mathcal{O}_Z$ that kill $\mathcal{N}^{vec}$ is a constant $\mathbb{F}_q$-étale sheaf of rank d. Therefore the sheaf of left $\mathcal{O}_{F^r,Z}$-module morphisms from $\mathcal{N}$ to $\mathcal{O}_Z$ that kill $\mathcal{N}^{vec}$ is contained in $\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{N}, \mathcal{O}_Z)^{con}$. For the reverse inclusion, it suffices to observe that any global homomorphism

$$\Gamma(Z, \mathcal{N}) \to A \tag{4.4.25}$$

must kill the K-vector space $\Gamma(Z, \mathcal{N})^{vec}$. □

Proof of Theorem 4.4.15. It is clear that all sections of M' have open support. By Lemma 4.4.23, $\Gamma(Z, M')$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{F^r,Z}}(\mathcal{M}/\mathcal{M}^{vec}, \mathcal{O}_Z). \tag{4.4.26}$$

Thus the $\mathbb{F}_q$-dimension of $\Gamma(Z, M')$ is equal to the $\mathcal{O}_Z$-rank of $\mathcal{M}/\mathcal{M}^{vec}$. Theorem 4.4.5 applied to $\mathcal{M}_{|\{\eta\}}$ implies that the K-dimension of $\mathcal{M}_\eta$ is equal to the $\mathbb{F}_q$-dimension of $M'_{\overline{\eta}}$. The stalks M'_s and $M'_{\overline{\eta}}$ of M are both finite, and therefore M' is constructible.

Let $\mathcal{M}'' = \mathcal{H}om_{\mathbb{F}_q}(M', \mathcal{O}_Z)$. Lemmas 4.4.20 and 4.4.23 imply isomorphisms

$$\mathcal{M}''^{vec} \cong \mathcal{H}om_{\mathbb{F}_q}\left(\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{M}^{vec}, \mathcal{O}_Z), \mathcal{O}_Z\right) \tag{4.4.27}$$

and

$$\mathcal{M}''/\mathcal{M}''^{vec} \cong \mathcal{H}om_{\mathbb{F}_q}\left(\mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{M}/\mathcal{M}^{vec}, \mathcal{O}_Z), \mathcal{O}_Z\right). \tag{4.4.28}$$

By Theorem 4.4.5 (applied after restriction to $\{\eta\}$), the double-dual homomorphism

$$\mathcal{M}^{vec} \to \mathcal{M}''^{vec} \tag{4.4.29}$$

is an isomorphism. The double-dual homomorphism

$$\mathcal{M}/\mathcal{M}^{vec} \to \mathcal{M}''/\mathcal{M}''^{vec} \tag{4.4.30}$$

is easily seen to be an isomorphism (since $\mathcal{M}/\mathcal{M}^{vec} \cong \mathcal{O}_Z^{\oplus d}$ for some $d \geq 0$). Therefore the double-dual homomorphism

$$\mathcal{M} \to \mathcal{M}'' \tag{4.4.31}$$

is an isomorphism by the 5-lemma. The proof of the first half of Theorem 4.4.15 is complete.

It is clear that the sheaf

$$\mathcal{M}' = \mathcal{H}om_{\mathbb{F}_q}\left(M, \mathcal{O}_Z\right) \tag{4.4.32}$$

is a torsion-free $\mathcal{O}_Z$-module. By Lemma 4.4.23, the sheaf $\mathcal{M}'/\mathcal{M}'^{vec}$ is isomorphic to

$$\mathcal{H}om_{\mathbb{F}_q}\left(M^{con}, \mathcal{O}_Z\right). \tag{4.4.33}$$

Therefore $\mathcal{M}'/\mathcal{M}'^{vec}$ is isomorphic to $\mathcal{O}_Z^{\oplus d}$, where d is the $\mathbb{F}_q$-rank of $\Gamma(Z, M)$. Theorem 4.4.5 applied to $M_{|\{\eta\}}$ implies that $\mathcal{M}'_{|\{\eta\}}$ is a finite-unit $\mathcal{O}_{F^r,\{\eta\}}$-module and that the K-dimension of $\mathcal{M}'_\eta$ is equal to the $\mathbb{F}_q$-dimension of $M_{\overline{\eta}}$.

Consider the exact sequence

$$0 \to \mathcal{M}'^{vec} \to \mathcal{M}' \to \mathcal{M}'/\mathcal{M}'^{vec} \to 0. \tag{4.4.34}$$

Since the sheaves $\mathcal{M}'^{vec}$ and $\mathcal{M}'/\mathcal{M}'^{vec}$ are quasi-coherent, the sheaf $\mathcal{M}'$ is quasi-coherent. Since the structural morphisms for $\mathcal{M}'^{vec}$ and $\mathcal{M}'/\mathcal{M}'^{vec}$ are isomorphisms, the structural morphism for $\mathcal{M}'$ is an isomorphism by the 5-lemma. Therefore $\mathcal{M}'$ is a unit $\mathcal{O}_{F^r,Z}$-module. Corollary 4.1.40 implies that $\mathcal{M}'$ is finite-unit.

Let $M'' = \mathcal{H}om_{\mathcal{O}_{F^r,Z}}(\mathcal{M}', \mathcal{O}_Z)$. Lemmas 4.4.20 and 4.4.23 imply isomorphisms

$$M''^{con} \cong \mathcal{H}om_{\mathcal{O}_{F^r,Z}}\left(\mathcal{H}om_{\mathbb{F}_q}\left(M^{con}, \mathcal{O}_Z\right), \mathcal{O}_Z\right) \tag{4.4.35}$$

and

$$M''/M''^{con} \cong \mathcal{H}om_{\mathcal{O}_{F^r,Z}}\left(\mathcal{H}om_{\mathbb{F}_q}\left(M/M^{con}, \mathcal{O}_Z\right), \mathcal{O}_Z\right). \tag{4.4.36}$$

By Theorem 4.4.5 (applied after restriction to $\{\eta\}$), the double-dual homomorphism

$$M/M^{con} \to M''/M''^{con} \tag{4.4.37}$$

is an isomorphism. It is easily seen that the double-dual homomorphism

$$M^{con} \to M''^{con} \tag{4.4.38}$$

for the constant sheaf M^{con} is an isomorphism. Therefore the double-dual homomorphism

$$M \to M'' \tag{4.4.39}$$

is an isomorphism by the 5-lemma. □

4.5 Examples

Example 4.5.1 Let M be the constant $\mathbb{F}_q$-étale sheaf $\underline{\mathbb{F}_q}$ on Spec A. Let

$$W = \mathrm{Hom}_{\mathbb{F}_q}(M, \mathcal{O}_{\mathrm{Spec}\ A}). \tag{4.5.2}$$

There is an isomorphism

$$W \to A \tag{4.5.3}$$

which identifies $w \in W$ with the image of $1 \in \mathbb{F}_q$ under w. This isomorphism is a left $A[F^r]$-module isomorphism. The only root of W is W itself (Proposition 4.3.16). The minimal root index of W is 0.

Example 4.5.4 Suppose that p is an odd prime. Consider the scheme morphism

$$\phi\colon \text{Spec } k[t'] \to \text{Spec } k[t] \tag{4.5.5}$$

determined by the k-algebra homomorphism $k[t] \to k[t']$ which sends t to $(t')^2$. Let A' be the stalk of $\mathcal{O}_{\text{Spec } k[t']}$ at (t'). Let $A \hookrightarrow A'$ be the stalk map determined by ϕ. The homomorphism $A \hookrightarrow A'$ determines a homomorphism of fraction fields $K \hookrightarrow K'$.

The field extension K'/K is a Galois field extension of order 2. There exists a nontrivial K-automorphism $\sigma\colon K' \to K'$ which maps t' to $-t'$. Define a homomorphism

$$\phi\colon \text{Gal}(K'/K) \to \mathbb{F}_q^* = \text{GL}(\mathbb{F}_q) \tag{4.5.6}$$

by $\phi(\sigma) = -1$. This homomorphism determines an $\mathbb{F}_q$-étale sheaf N on Spec K. Let $M = j_!N$, where $j\colon \text{Spec } K \to \text{Spec } A$ denotes inclusion.

Let W be the finite-unit $A[F^r]$-module

$$\text{Hom}_{\mathbb{F}_q}\left(M, \mathcal{O}_{\text{Spec } A}\right). \tag{4.5.7}$$

By Theorem 4.4.15, W is a one-dimensional K-vector space. Let a be a nonzero element of the one-dimensional $\mathbb{F}_q$-vector space $M(\text{Spec } K')$. Define an $\mathbb{F}_q$-étale sheaf morphism

$$w\colon M \to \mathcal{O}_{\text{Spec } A} \tag{4.5.8}$$

by

$$w(\text{Spec } K')(a) = t'. \tag{4.5.9}$$

The element $w \in W$ generates W as a K-vector space. The element $F^r w \in W$ is the $\mathbb{F}_q$-morphism $M \to \mathcal{O}_{\text{Spec } A}$ defined by

$$(F^r w)(\text{Spec } K')(a) = (t')^q. \tag{4.5.10}$$

Thus $F^r(w) = t^{(q-1)/2} w$.

The finitely-generated A-submodules of W are all of the form $A(t^n w) \subseteq W$, where n is an integer. Note that

$$F^r(t^n w) = t^{nq + \frac{q-1}{2}} w. \tag{4.5.11}$$

If $A(t^n w) \subseteq W$ is a root for W, the inequality

$$nq + \frac{q-1}{2} \leq n \tag{4.5.12}$$

must be satisfied. The highest value of n which makes this inequality hold is $n = -1$. Let $W_0 = (t^{-1} w)$. For any $m \geq 1$,

$$F^{rm}(t^{-1} w) = t^{-q^m + \frac{q-1}{2} \cdot (q^{m-1} + q^{m-2} + \ldots + 1)}. \tag{4.5.13}$$

The exponent in the above equation tends to $-\infty$ as m tends to ∞. Therefore W_0 generates W as a left $A[F^r]$-module. We conclude that W_0 is the minimal root for W.

Let W_1 be the A-module generated by

$$F^r(t^{-1} w) = t^{\frac{-q-1}{2}} w \in W. \tag{4.5.14}$$

The k-dimension of W_1/W_0 is $(q-1)/2$, therefore

$$\text{ind } W_0 = \frac{(q-1)/2}{q-1} = \frac{1}{2}. \tag{4.5.15}$$

The minimal root index of W is $\frac{1}{2}$.

Chapter 5

The Riemann-Hilbert correspondence on a curve

This chapter extends Theorem 4.4.15 to a correspondence on smooth k-curves. As in previous chapters, let r be a positive integer and let $q = p^r$.

5.1 The functor $\mathcal{H}om_{\mathbb{F}_q}(\cdot, \mathcal{O}_X)$

Let X be a smooth k-curve. Recall from Section 2.1 that if x is a closed point of X and M is an $\mathbb{F}_q$-étale sheaf on X, then $M_{(x)}$ denotes the pullback of M via the morphism

$$\text{Spec } \mathcal{O}_{X,x} \to X. \tag{5.1.1}$$

Proposition 5.1.2 *Let M be an object of* $\mathbf{Mod}^c(X, \mathbb{F}_q)$. *Let x be a closed point of X. The natural homomorphism*

$$\mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_X)_x \to \text{Hom}_{\mathbb{F}_q}\left(M_{(x)}, \mathcal{O}_{\text{Spec } \mathcal{O}_{X,x}}\right) \tag{5.1.3}$$

is an isomorphism.

We will prove Proposition 5.1.2 by reduction to the following special case.

Proposition 5.1.4 *Let Y be a smooth affine curve over k. Let $Z \to Y$ be a finite Galois cover which is totally ramified at one closed point $y \in |Y|$ and unramified elsewhere. Let $\{z\}$ be the pre-image of $\{y\}$ in Z, and let $Z' \subseteq Z$ be the complement of $\{z\}$. Let N be a constructible $\mathbb{F}_q$-étale sheaf on Y such that $N_{|Z'}$ is constant. Then the natural homomorphism*

$$\mathcal{H}om_{\mathbb{F}_q}\left(N, \mathcal{O}_Y\right)_y \to \mathrm{Hom}_{\mathbb{F}_q}\left(N_{(y)}, \mathcal{O}_{\mathrm{Spec}\ \mathcal{O}_{Y,y}}\right) \tag{5.1.5}$$

is an isomorphism.

Proof. The curves Y, Z, and Z' are all affine. Let $Y = \mathrm{Spec}\ R$, $Z = \mathrm{Spec}\ S$, and $Z' = \mathrm{Spec}\ S'$. If $\mathrm{Spec}\ Q \to \mathrm{Spec}\ R$ is any étale morphism, then morphisms

$$N_{|\mathrm{Spec}\ Q} \to \mathcal{O}_{\mathrm{Spec}\ Q} \tag{5.1.6}$$

may be expressed as commutative diagrams

$$\begin{array}{ccc} N(\mathrm{Spec}\ S') & \xrightarrow{\phi} & S' \otimes_R Q \\ \uparrow{\scriptstyle \rho_1} & & \uparrow{\scriptstyle \rho_2} \\ N(\mathrm{Spec}\ R) & \xrightarrow{\psi} & Q \end{array} \tag{5.1.7}$$

in which ρ_1 and ρ_2 are sheaf restriction maps, ϕ and ψ are $\mathbb{F}_q$-linear homomorphisms, and ϕ is $\mathrm{Aut}(S'/R)$-equivariant. Similarly, morphisms

$$N_{(y)} \to \mathcal{O}_{\mathrm{Spec}\ \mathcal{O}_{Y,y}} \tag{5.1.8}$$

may be expressed as commutative diagrams

$$\begin{CD} N(\text{Spec } S') @>>> S' \otimes_R \mathcal{O}_{Y,y} \\ @AAA @AAA \\ N(\text{Spec } R) @>>> \mathcal{O}_{Y,y} \end{CD} \tag{5.1.9}$$

in which the vertical maps are sheaf restriction maps, the horizontal maps are $\mathbb{F}_q$-linear homomorphisms, and the top map is $\text{Aut}(S'/R)$-equivariant.

Suppose that

$$\begin{CD} N(\text{Spec } S') @>>> S' \otimes_R \mathcal{O}_{Y,y} \\ @AAA @AAA \\ N(\text{Spec } R) @>>> \mathcal{O}_{Y,y} \end{CD} \tag{5.1.10}$$

is the diagram for a morphism $N_{(y)} \to \mathcal{O}_{\text{Spec } \mathcal{O}_{Y,y}}$. Since $N(\text{Spec } R)$ and $N(\text{Spec } S')$ are finite, there exists an étale R-algebra $P \subseteq \mathcal{O}_{Y,y}$ such that the images of $N(\text{Spec } R)$ and $N(\text{Spec } S')$ are contained in P and $S' \otimes_R P$, respectively. Thus (5.1.10) determines a commutative diagram

$$\begin{CD} N(\text{Spec } S') @>>> S' \otimes_R P \\ @AAA @AAA \\ N(\text{Spec } R) @>>> P. \end{CD} \tag{5.1.11}$$

We conclude that any $\mathbb{F}_q$-linear morphism $N_{(y)} \to \mathcal{O}_{\text{Spec } \mathcal{O}_{Y,y}}$ extends to an $\mathbb{F}_q$-linear morphism from $N \to \mathcal{O}_Y$ on some étale neighborhood of y. We have constructed an inverse to homomorphism (5.1.5). $\square$

Proof of Proposition 5.1.2. Replacing X with an open subcurve if necessary, we may assume that X is affine and that the constructible sheaf M is locally constant on $X \smallsetminus \{x\}$. Let $Z' \to X \smallsetminus \{x\}$ be a finite Galois étale cover such that $M_{|Z'}$ is constant.

Let $K_0/K(X)$ be the largest subextension of $K(Z')/K(X)$ that is unramified at x. The tower of field extensions

$$K(Z') \supseteq K_0 \supseteq K(X) \tag{5.1.12}$$

determines a diagram of smooth projective curves,

$$\begin{array}{ccc} \overline{Z'} & \longrightarrow & W \\ & \searrow & \downarrow \\ & & \overline{X} \end{array} \tag{5.1.13}$$

where $\overline{Z'}$ and $\overline{X}$ are the smooth projective closures of Z' and X, respectively, and W is the unique smooth projective curve over k whose fraction field is K_0. Let

$$\begin{array}{ccc} \overline{Z'} \times_{\overline{X}} X & \longrightarrow & W \times_{\overline{X}} X \\ & \searrow & \downarrow \\ & & X \end{array} \tag{5.1.14}$$

be the diagram obtained from (5.1.13) via base change. The morphism

$$\overline{Z'} \times_{\overline{X}} X \to X \tag{5.1.15}$$

is Galois and finite, and étale away from x. The morphism

$$W \times_{\overline{X}} X \to X \tag{5.1.16}$$

is Galois and finite and étale at all points of X. The morphism

$$\overline{Z'} \times_{\overline{X}} X \to W \times_{\overline{X}} X \tag{5.1.17}$$

is Galois and finite, étale away from x, and totally ramified at x. Proposition 5.1.4 may therefore be applied with $Y = W \times_{\overline{X}} X$, $Z = \overline{Z'} \times_{\overline{X}} X$, and $N = M_{|W \times_{\overline{X}} X}$. This application completes the current proof, since the assertion of Proposition 5.1.2 is local at x. $\square$

Proposition 5.1.18 *Let M be a constructible $\mathbb{F}_q$-étale sheaf on X whose sections all have open support. Let*

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_X), \tag{5.1.19}$$

with left $\mathcal{O}_{F^r,X}$-module structure given by the left $\mathcal{O}_{F^r,X}$-module structure on $\mathcal{O}_X$. Then $\mathcal{M}$ is a torsion-free finite-unit $\mathcal{O}_{F^r,X}$-module.

Proof. The proof of this proposition consists of three lemmas.

Lemma 5.1.20 *The $\mathcal{O}_X$-module $\mathcal{M}$ is quasi-coherent.*

Proof. Let $U \subseteq X$ be a nonempty open subset on which M is locally constant. Let $j : U \to X$ be the inclusion morphism. The sheaf

$$\mathcal{H}om_{\mathbb{F}_q}\left(M_{|U}, \mathcal{O}_U\right) \tag{5.1.21}$$

is locally free of finite rank as an $\mathcal{O}_U$-module. The pushforward

$$j_*\mathcal{H}om_{\mathbb{F}_q}\left(M_{|U}, \mathcal{O}_U\right) \cong \mathcal{H}om_{\mathbb{F}_q}(M, j_*\mathcal{O}_U) \tag{5.1.22}$$

is a quasi-coherent $\mathcal{O}_X$-module. There is a natural morphism

$$\mathcal{M} \hookrightarrow \mathcal{H}om_{\mathbb{F}_q}(M, j_*\mathcal{O}_U). \tag{5.1.23}$$

To show that $\mathcal{M}$ is quasi-coherent, it suffices to show that the cokernel of this morphism is quasi-coherent.

The image of (5.1.23) consists of the morphisms $M \to j_*\mathcal{O}_U$ that map M_x into $\mathcal{O}_{X,x}$ for each $x \in |X \smallsetminus U|$. Suppose that x is an element of $|X \smallsetminus U|$, and suppose that

ϕ is a morphism from M to $j_*\mathcal{O}_U$ over a Zariski open neighborhood of x. Choose a local parameter t at x. Since M_x is finite, we may choose n sufficiently large so that $t^n\phi$ maps M_x into $\mathcal{O}_{X,x}$. We conclude that the cokernel of (5.1.23) is a quasi-coherent skyscraper sheaf supported at $|X \smallsetminus U|$. This completes the proof. □

Lemma 5.1.24 *The structural morphism*

$$F_X^*\mathcal{M} \to \mathcal{M} \tag{5.1.25}$$

is an isomorphism.

Proof. It suffices to show that for any closed point $x \in |X|$, the structural morphism of $\mathcal{M}_x$ is an isomorphism. By Proposition 5.1.2, there exist isomorphisms

$$\mathcal{M}_x \to \mathrm{Hom}_{\mathbb{F}_q}\left(M_{(x)}, \mathcal{O}_{\mathrm{Spec}\ \mathcal{O}_{X,x}}\right) \tag{5.1.26}$$

for each closed point $x \in |X|$. By Theorem 4.4.15, each left $\mathcal{O}_{X,x}[F^r]$-module

$$\mathrm{Hom}_{\mathbb{F}_q}\left(M_{(x)}, \mathcal{O}_{\mathrm{Spec}\ \mathcal{O}_{X,x}}\right) \tag{5.1.27}$$

is a unit $\mathcal{O}_{X,x}[F^r]$-module. □

Lemma 5.1.28 *The left $\mathcal{O}_{F^r,X}$-module $\mathcal{M}$ is generated by a finite number of sections.*

Proof. Let $U \subseteq X$ be a Zariski open subset on which M is locally constant. By Proposition 5.1.2 and Theorem 4.4.15, each stalk $\mathcal{M}_x$ is a finite-unit $\mathcal{O}_{X,x}[F^r]$-module. For each point $x \in |X \smallsetminus U|$, choose a finite set of sections of $\mathcal{M}$ on a Zariski open neighborhood of x which generate $\mathcal{M}_x$ as a left $\mathcal{O}_{X,x}[F^r]$-module. Choose a finite set of sections of the coherent $\mathcal{O}_U$-module $\mathcal{M}_{|U}$ which generate $\mathcal{M}_{|U}$ as an $\mathcal{O}_U$-module. Let $\mathcal{M}' \subseteq \mathcal{M}$ be the sub-left-$\mathcal{O}_{F^r,X}$-module generated by all of the aforementioned sections. The stalk of $\mathcal{M}'$ at any closed point x is equal to $\mathcal{M}$. Therefore $\mathcal{M}' = \mathcal{M}$. □

It is clear that $\mathcal{M}$ is torsion-free as an $\mathcal{O}_X$-module. The proof of Proposition 5.1.18 is complete. □

5.2 The functor $\mathcal{H}om_{\mathcal{O}_{F^r,X}}(\cdot, \mathcal{O}_X)$

Let X be a smooth k-curve.

Proposition 5.2.1 *Let $\mathcal{M}$ be an object of $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r,X})$. Let x be a closed point of X. The natural homomorphism*

$$\mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X)_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}[F^r]}(\mathcal{M}_x, \mathcal{O}_{X,x}) \tag{5.2.2}$$

is an isomorphism.

Proof. It is clear that (5.2.2) is injective. To prove the proposition it suffices to show that any element of

$$\mathrm{Hom}_{\mathcal{O}_{X,x}[F^r]}(\mathcal{M}_x, \mathcal{O}_{X,x}) \tag{5.2.3}$$

may be extended to a left $\mathcal{O}_{F^r,X}$-module homomorphism from $\mathcal{M}$ to $\mathcal{O}_X$ over an étale neighborhood of x.

Let

$$\phi\colon \mathcal{M}_x \to \mathcal{O}_{X,x} \tag{5.2.4}$$

be a left $\mathcal{O}_{X,x}[F^r]$-module homomorphism. Let $U \subseteq X$ be an affine neighborhood of x. Let $R = \Gamma(U, \mathcal{O}_U)$ and $P = \Gamma(U, \mathcal{M})$. Let $\{p_1, \ldots, p_c\} \subseteq P$ be a subset which generates P as a left $R[F^r]$-module. Choose an étale R-algebra $R' \subseteq \mathcal{O}_{X,x}$ large enough to contain the images of the stalks of each p_i under ϕ. The homomorphism

$$\mathcal{O}_{X,x} \otimes_R P \to \mathcal{O}_{X,x} \tag{5.2.5}$$

determined by ϕ restricts to a homomorphism

$$R' \otimes_R P \to R'. \tag{5.2.6}$$

Thus there is a homomorphism from $\mathcal{M}_{|\text{Spec } R'}$ to $\mathcal{O}_{\text{Spec } R'}$ whose stalk is ϕ. □

Proposition 5.2.7 *Let $\mathcal{M}$ be an object of* $\mathbf{LMod}^{fu}(X, \mathcal{O}_{F^r,X})$. *Let*

$$M = \mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X). \tag{5.2.8}$$

Then there exists a nonempty étale X-scheme V such that $M_{|V}$ is a constant $\mathbb{F}_q$-sheaf of finite rank.

Proof. By Corollary 3.3.11, there exists a nonempty open subset $X' \subseteq X$ such that $\mathcal{M}_{|X'}$ a coherent $\mathcal{O}_{X'}$-module. Let $\alpha \in |X'|$ denote the generic point, and let

$$\overline{\alpha}\colon \text{Spec } \overline{k(\alpha)} \to X' \tag{5.2.9}$$

denote a geometric point at α. By Proposition 3.4.11, the geometric stalk $\mathcal{M}_{\overline{\alpha}}$ has a $\overline{k(\alpha)}$-basis that is fixed by F^r. Choose an étale scheme U over X' on which there exist representatives

$$m_1, \ldots, m_e \in \mathcal{M}(U) \tag{5.2.10}$$

for the elements of this basis. The coherent subsheaf of $\mathcal{M}_{|U}$ generated by $\{m_i\}_{i=1}^e$ has the same generic rank as $\mathcal{M}_{|U}$. Let $V \subseteq U$ be a nonempty open subset on which these two sheaves are equal. Then $\mathcal{M}_{|V}$ is isomorphic as a left $\mathcal{O}_{F^r,X}$-module to $\mathcal{O}_V^{\oplus e}$. Therefore $M_{|V}$ is isomorphic to a constant $\mathbb{F}_q$-étale sheaf of rank e. □

Corollary 5.2.11 *Let $\mathcal{M}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,X}$-module. Then*

$$M = \mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X) \tag{5.2.12}$$

is a constructible $\mathbb{F}_q$-étale sheaf on X.

Proof. Proposition 5.2.7 implies that M is locally constant on a nonempty open subset of X. Theorem 4.4.15 implies (via Proposition 5.2.1) that the stalks of M are finite. □

Proposition 5.2.13 *Let $\mathcal{M}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,X}$-module. The double-dual homomorphism*

$$\mathcal{M} \to \mathcal{H}om_{\mathbb{F}_q}\left(\mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X), \mathcal{O}_X\right) \tag{5.2.14}$$

is an isomorphism.

Proposition 5.2.15 *Let M be a constructible $\mathbb{F}_q$-étale sheaf on X whose sections all have open support. The double-dual homomorphism*

$$M \to \mathcal{H}om_{\mathcal{O}_{F^r,X}}\left(\mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_X), \mathcal{O}_X\right) \tag{5.2.16}$$

is an isomorphism.

Proof of Propositions 5.2.13 and 5.2.15. It suffices to show that morphisms (5.2.14) and (5.2.16) induce isomorphisms on closed stalks. This assertion follows from Theorem 4.4.15 via Propositions 5.1.2 and 5.2.1. □

5.3 Roots on curves

In this section we develop the properties of roots on curves in preparation for the proof of Theorem 5.4.9. Again let X be a smooth k-curve.

Let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root of $\mathcal{M}$. Let

$$\Lambda_{\mathcal{M}} \colon \mathcal{M} \to F_X^* \mathcal{M} \tag{5.3.1}$$

denote the inverse of the structural morphism for $\mathcal{M}$. By Property 2 of Definition 3.3.1, the image of $\mathcal{M}_0$ under Λ is contained in the subsheaf $F_X^*\mathcal{M}_0 \subseteq F_X^*\mathcal{M}$. Let

$$\lambda_{\mathcal{M}_0} \colon \mathcal{M}_0 \to F_X^* \mathcal{M}_0 \tag{5.3.2}$$

denote the restriction of Λ. Note that if $U \subseteq X$ is a Zariski open subset such that $\mathcal{M}_{|U} = (\mathcal{M}_0)_{|U}$ (recall Proposition 3.3.9), then the restrictions of $\Lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{M}_0}$ to U are identical.

We describe a canonical left $\mathcal{O}_{F^r,X}$-module structure on the coherent sheaf dual

$$\mathcal{M}_0^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_0, \mathcal{O}_X) \tag{5.3.3}$$

of $\mathcal{M}_0$. Let $V \to X$ be an étale morphism, and let

$$\phi \colon\ (\mathcal{M}_0)_{|V} \to \mathcal{O}_V \tag{5.3.4}$$

be an $\mathcal{O}_V$-module homomorphism. Then $F^r(\phi)$ is the composition of the diagram

$$\begin{array}{ccc} \mathcal{M}_{0|V} & & \mathcal{O}_V \\ \downarrow{\scriptstyle \lambda_{\mathcal{M}_0}} & & \uparrow{\scriptstyle \cong} \\ F_V^{r*}\left(\mathcal{M}_{0|V}\right) & \xrightarrow{F_V^{r*}\phi} & F_V^{r*}\mathcal{O}_V. \end{array} \tag{5.3.5}$$

If V is affine and $m \in \mathcal{M}_0(V)$, and

$$m = \sum_{j=1}^{n} f_j F^r(m_j) \tag{5.3.6}$$

with $f_j \in \mathcal{O}_X(V)$ and $m_j \in \mathcal{M}_0(V)$, then

$$(F^r\phi)\, m = \sum_{j=1}^{n} f_j F^r(\phi(m_j)). \tag{5.3.7}$$

Lemma 5.3.8 *Let $\mathcal{M}$ be a finite-unit $\mathcal{O}_{F^r,X}$-module, and let $\mathcal{M}_0$ be a root of $\mathcal{M}$. Under the canonical left $\mathcal{O}_{F^r,X}$-module structure on $\mathcal{M}_0^\vee$, the action of F^r on $\mathcal{M}_0^\vee$ is injective.*

Proof. Let V be a connected étale X-scheme, and let

$$\phi\colon\ (\mathcal{M}_0)_{|V} \to \mathcal{O}_V \tag{5.3.9}$$

be a nonzero morphism. By Proposition 3.3.9, we may choose a nonempty open subset $U \subseteq X$ such that $(\mathcal{M}_0)_{|U} = \mathcal{M}_{|U}$. Let $V' = V \times_X U$. The restriction

$$\phi_{|V'}\colon\ (\mathcal{M}_0)_{|V'} \to \mathcal{O}_{V'} \tag{5.3.10}$$

is also nonzero. The element $F^r(\phi)_{|V'} = F^r(\phi_{|V'}) \in \mathcal{M}_0^\vee(V')$ is the composition of the diagram

$$\begin{array}{ccc} \mathcal{M}_{0|V'} & & \mathcal{O}_{V'} \\ \Big\downarrow \lambda_{\mathcal{M}_0} & & \cong \Big\uparrow \\ F^{r*}_{V'}\left(\mathcal{M}_{0|V'}\right) & \xrightarrow{F^{r*}_{V'}\phi} & F^{r*}_{V'}\mathcal{O}_{V'}. \end{array} \tag{5.3.11}$$

The vertical morphisms are both isomorphisms and the horizontal morphism is nonzero. Therefore $F^r(\phi)_{|V'}$ is nonzero, and likewise $F^r(\phi)$ is nonzero. □

Proposition 5.3.12 *Let $\mathcal{M}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,X}$-module, and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root for $\mathcal{M}$. Then the restriction morphism*

$$\mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_0, \mathcal{O}_X) \tag{5.3.13}$$

is injective, and its image is the subsheaf of F^r-invariant sections of $\mathcal{M}_0^\vee$.

Proof. The fact that each section of the image of (5.3.13) is F^r-invariant is apparent from the definition of the left $\mathcal{O}_{F^r,X}$-module structure of $\mathcal{M}_0^\vee$. We construct an inverse morphism

$$\left(\mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{M}_0, \mathcal{O}_X\right)\right)^{F^r} \to \mathcal{H}om_{\mathcal{O}_{F^r,X}}\left(\mathcal{M}, \mathcal{O}_X\right). \tag{5.3.14}$$

Let $U \to X$ be an étale morphism, and let

$$\psi\colon \left(\mathcal{M}_0\right)_{|U} \to \mathcal{O}_U \tag{5.3.15}$$

be an F^r-invariant morphism. Let

$$\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots \tag{5.3.16}$$

be the filtration for $\mathcal{M}$ described in Proposition 3.3.4. For each $n \geq 0$, let

$$\psi_n\colon \left(\mathcal{M}_n\right)_{|U} \to \mathcal{O}_U \tag{5.3.17}$$

be the composition of the diagram

$$\begin{array}{ccc} \mathcal{M}_{n|U} & & \mathcal{O}_U \\ \downarrow \cong & & \cong \uparrow \\ F_U^{rn*}\left(\mathcal{M}_{0|U}\right) & \xrightarrow{F^{rn*}\psi} & F_U^{rn*}\mathcal{O}_U \end{array} \tag{5.3.18}$$

Since ψ is F^r-invariant, the morphisms ψ_n are compatible and thus determine an F^r-invariant morphism $\mathcal{M} \to \mathcal{O}_X$ which extends ψ. □

Proposition 5.3.19 *Let M be a constructible $\mathbb{F}_q$-étale sheaf on X whose sections all have open support. Let*

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_q}\left(M, \mathcal{O}_X\right), \tag{5.3.20}$$

and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root for $\mathcal{M}$. The double-dual homomorphism

$$M \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_0, \mathcal{O}_X) = \mathcal{M}_0^\vee \tag{5.3.21}$$

fits into an exact sequence

$$0 \longrightarrow M \longrightarrow \mathcal{M}_0^\vee \xrightarrow{1-F^r} \mathcal{M}_0^\vee \longrightarrow 0. \tag{5.3.22}$$

Proof. The morphism $M \to \mathcal{M}_0^\vee$ is equal to the composition

$$M \to \mathcal{H}om_{\mathcal{O}_{F^r,X}}(\mathcal{M}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_0, \mathcal{O}_X), \tag{5.3.23}$$

where the first morphism is a double-dual morphism and the second morphism is given by restriction. The first morphism is an isomorphism (by Proposition 5.2.15), and the second morphism is an injective morphism whose image is the kernel of $1 - F^r$ (by Proposition 5.3.12).

To show that the action of $1 - F^r$ on $\mathcal{M}_0^\vee$ is surjective, it suffices to show that the action of $1 - F^r$ on $(\mathcal{M}_0^\vee)_x$ is surjective for every closed point $x \in |X|$. This fact is given by Proposition 4.1.1. □

Proposition 5.3.24 *Let $\mathcal{M}$ be a torsion-free finite-unit $\mathcal{O}_{F^r,X}$-module. There exists a root $\mathcal{M}_0 \subseteq \mathcal{M}$ which is contained in all other roots of $\mathcal{M}$. For each closed point $x \in |X|$, the stalk $(\mathcal{M}_0)_x$ is the minimal root of $\mathcal{M}_x$.*

Proof. For any étale morphism $V \to X$, let $\mathcal{M}_0(V) \subseteq \mathcal{M}(V)$ be the subset consisting of sections $m \in \mathcal{M}(V)$ such that for any closed point $x \in |X|$ and any diagram

$$\begin{array}{ccc} \text{Spec } k(x) & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X, \end{array} \tag{5.3.25}$$

the stalk element at x represented by m is contained in the minimal root of $\mathcal{M}_x$.

Lemma 5.3.26 *For any closed point $x \in |X|$, $(\mathcal{M}_0)_x$ is equal to the minimal root of $\mathcal{M}_x$.*

Proof. Let x be a closed point of X. Let $m_x \in \mathcal{M}_x$ be an element that is contained in the minimal root of $\mathcal{M}_x$. Choose an étale neighborhood

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X, \end{array} \tag{5.3.27}$$

and a section $m \in \mathcal{M}(V)$ which represents m_x. By Corollary 3.3.11, there exists a nonempty open subset $U \subseteq X$ on which $\mathcal{M}$ is coherent. Since $\mathcal{M}$ is torsion-free, $\mathcal{M}_{|U}$ is a locally free $\mathcal{O}_U$-module of finite rank. By Proposition 4.3.16, for any closed point $u \in |U|$, the minimal root of $\mathcal{M}_u$ is $\mathcal{M}_u$. Let $x' \in |V|$ be the closed point which is the image of Spec k in (5.3.27). Then by construction, the restriction

$$m_{|(V \times_X U) \cup \{x'\}} \in \mathcal{M}((V \times_X U) \cup \{x'\}) \tag{5.3.28}$$

is contained in the subsheaf $\mathcal{M}_0$. Thus the stalk element m_x is contained in $(\mathcal{M}_0)_x$.

We have shown that the minimal root of $\mathcal{M}_x$ is contained in $(\mathcal{M}_0)_x$. The reverse inclusion is obvious. □

Lemma 5.3.29 *The subsheaf $\mathcal{M}_0 \subseteq \mathcal{M}$ is a root for $\mathcal{M}$.*

Proof. Properties 2 and 3 of Definition 3.3.1 follow easily from the corresponding properties for local roots. It is necessary only to show that $\mathcal{M}_0$ is a coherent $\mathcal{O}_X$-module.

By Corollary 3.3.11, there exists a nonempty open subset $U \subseteq X$ on which $\mathcal{M}$ is coherent. Since $\mathcal{M}$ is torsion-free, $\mathcal{M}_{|U}$ is a locally free $\mathcal{O}_U$-module of finite rank. For any closed point $u \in |U|$, the minimal root of $\mathcal{M}_u$ is $\mathcal{M}_u$ (by Proposition 4.3.16). Thus $(\mathcal{M}_0)_{|U} = \mathcal{M}_{|U}$.

Let x be a closed point of X that lies outside of U. The root $(\mathcal{M}_0)_x \subseteq \mathcal{M}_x$ spans $\mathcal{M}_x$ (Lemma 4.3.12 of Theorem 4.3.1). Thus

$$\mathcal{M}_x / (\mathcal{M}_0)_x \cong (\mathcal{M}/\mathcal{M}_0)_x \tag{5.3.30}$$

is a torsion $\mathcal{O}_{X,x}$-module. We conclude that $\mathcal{M}/\mathcal{M}_0$ is a quasi-coherent skyscraper sheaf. Since $\mathcal{M}$ is quasi-coherent and $\mathcal{M}/\mathcal{M}_0$ is quasi-coherent, the sheaf $\mathcal{M}_0$ is quasi-coherent.

For each $x \in |X \smallsetminus U|$, there exists a finite $\mathcal{O}_{X,x}$-module generating set for $(\mathcal{M}_0)_x$, which may be represented by a finite set of sections of $\mathcal{M}_0$ on some étale neighborhood of x. The sheaf $(\mathcal{M}_0)_{|U} = \mathcal{M}_{|U}$ is generated as an $\mathcal{O}_U$-module by a finite set of sections. Let $\mathcal{M}_0' \subseteq \mathcal{M}_0$ be the $\mathcal{O}_X$-submodule of $\mathcal{M}_0$ generated by all of the aforementioned sections. Then $(\mathcal{M}_0')_x = (\mathcal{M}_0)_x$ for any closed point x, and thus $\mathcal{M}_0' = \mathcal{M}_0$. We conclude that $\mathcal{M}_0$ is a coherent $\mathcal{O}_X$-module. □

Lemma 5.3.31 *Any root of $\mathcal{M}$ contains $\mathcal{M}_0$.*

Proof. Let $\mathcal{M}' \subseteq \mathcal{M}$ be a root of $\mathcal{M}$. It is easily seen that $\mathcal{M}'_x$ is a root for $\mathcal{M}_x$ for every closed point $x \in |X|$. Thus $(\mathcal{M}_0)_x \subseteq \mathcal{M}'_x$ for every closed point $x \in |X|$. Therefore the subsheaf $\mathcal{M}_0$ is contained in $\mathcal{M}'$. □

The proof of Proposition 5.3.24 is complete. □

5.4 Cohomology and the Riemann-Hilbert correspondence

In this section, let Y be a smooth projective k-curve.

Let M be a constructible $\mathbb{F}_q$-étale sheaf whose sections all have open support. By Proposition 5.1.18, the sheaf

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_Y), \tag{5.4.1}$$

is a finite-unit $\mathcal{O}_{F^r,Y}$-module. Let $\mathcal{M}_0$ be a root for $\mathcal{M}$. The left $\mathcal{O}_{F^r,Y}$-module structure on the dual sheaf $\mathcal{M}_0^\vee$ described in Section 5.3 determines a left $k[F^r]$-module structure on the cohomology group $H^j(Y, \mathcal{M}_0^\vee)$ for each $j \geq 0$. Since Y is projective and $\mathcal{M}_0^\vee$ is coherent, each group $H^j(Y, \mathcal{M}_0^\vee)$ is a finite-dimensional k-vector space. The double-dual homomorphism $M \to \mathcal{M}_0^\vee$ determines an $\mathbb{F}_q$-linear homomorphism

$$H^j(Y, M) \to H^j\left(Y, \mathcal{M}_0^\vee\right) \tag{5.4.2}$$

for every $j \geq 0$.

Proposition 5.4.3 *Let M be a constructible $\mathbb{F}_q$-étale sheaf on Y whose sections all have open support. Let*

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_X), \tag{5.4.4}$$

and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a root for $\mathcal{M}$. For any $j \geq 0$, the sequence

$$0 \longrightarrow H^j(Y, M) \longrightarrow H^j(Y, \mathcal{M}_0^\vee) \xrightarrow{1-F^r} H^j(Y, \mathcal{M}_0^\vee) \longrightarrow 0 \tag{5.4.5}$$

is exact.

Proof. Proposition 5.3.19 asserts that the sequence

$$0 \longrightarrow M \longrightarrow \mathcal{M}_0^\vee \xrightarrow{1-F^r} \mathcal{M}_0^\vee \longrightarrow 0 \tag{5.4.6}$$

is exact. Since each cohomology group $H^j(Y, \mathcal{M}_0^\vee)$ is a finite-dimensional k-vector space, the maps

$$H^j(Y, \mathcal{M}_0^\vee) \xrightarrow{1-F^r} H^j(Y, \mathcal{M}_0^\vee) \tag{5.4.7}$$

are all surjective by Proposition 3.4.15. Therefore the long exact sequence of cohomology groups associated to (5.4.6) breaks up into short exact sequences. □

Let M be a constructible $\mathbb{F}_q$-étale sheaf on Y whose sections all have open support. Let $\chi(Y, M)$ and $\chi(Y, \mathcal{O}_Y)$ denote the Euler characteristics of M and $\mathcal{O}_Y$, respectively. If y is a closed point of Y, let $\mathfrak{C}(M_{(y)})$ denote the minimal root index of the finite-unit $\mathcal{O}_{Y,y}[F^r]$-module

$$\mathcal{H}om_{\mathbb{F}_q}\left(M_{(y)}, \mathcal{O}_{\mathrm{Spec}\ \mathcal{O}_{Y,y}}\right). \tag{5.4.8}$$

(See Section 4.3 for the definition of the minimal root index.)

Theorem 5.4.9 *Let Y be a smooth projective k-curve. Let M be a constructible $\mathbb{F}_q$-étale sheaf on Y whose sections all have open support. Let d be the generic rank of M. Then*

$$\chi(Y, M) \geq d \cdot \chi(Y, \mathcal{O}_Y) - \sum_{y \in Y(k)} \mathfrak{C}(M_{(y)}). \tag{5.4.10}$$

Proof. Let

$$\mathcal{M} = \mathcal{H}om_{\mathbb{F}_q}(M, \mathcal{O}_Y). \tag{5.4.11}$$

Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the minimal root for $\mathcal{M}$ (see Proposition 5.3.24). For each $y \in Y(k)$, the F^r-index of $(\mathcal{M}_0)_y$ in $\mathcal{M}_y$ is $\mathfrak{C}(M_{(y)})$.

Let

$$\lambda_{\mathcal{M}_0} \colon \mathcal{M}_0 \to F_Y^{r*}\mathcal{M}_0 \tag{5.4.12}$$

be the morphism determined by the structural morphism of $\mathcal{M}$ (see Section 5.3). The cokernel of $\lambda_{\mathcal{M}_0}$ is a skyscraper sheaf supported at the points y for which $\mathfrak{C}(M_{(y)}) > 0$. For each $y \in Y(k)$, the k-dimension of the stalk at y of the cokernel of $\lambda_{\mathcal{M}_0}$ is $(q-1)\mathfrak{C}(M_{(y)})$. Let $\deg \mathcal{M}_0$ and $\deg F_Y^{r*}\mathcal{M}_0$ denote the degrees of $\mathcal{M}_0$ and $F_Y^{r*}\mathcal{M}_0$ as coherent sheaves. Then

$$\begin{aligned}\deg \mathcal{M}_0 &= \deg F_Y^{r*}\mathcal{M}_0 - (q-1)\sum_{y\in Y(k)} \mathfrak{C}(M_{(y)}) && (5.4.13)\\ &= q\deg \mathcal{M}_0 - (q-1)\sum_{y\in Y(k)} \mathfrak{C}(M_{(y)}), && (5.4.14)\end{aligned}$$

and thus

$$\deg \mathcal{M}_0 = \sum_{y\in Y(k)} \mathfrak{C}(M_{(y)}). \qquad (5.4.15)$$

By Proposition 5.4.3, for each $j \geq 0$, $H^j(X, M)$ is isomorphic to the F^r-fixed subspace of $H^j(X, \mathcal{M}_0^\vee)$. By Proposition 3.4.13,

$$\dim_{\mathbb{F}_q} H^j(Y, M) \leq \dim_k H^j(Y, \mathcal{M}_0^\vee) \qquad (5.4.16)$$

for each $j \geq 0$. Equality occurs when $j = 0$, because the action of F^r on $\mathcal{M}_0^\vee$ (and likewise on $H^0(Y, \mathcal{M}_0^\vee)$) is injective (Lemma 5.3.8). When $j > 1$,

$$\dim_{\mathbb{F}_q} H^j(Y, M) = \dim_k H^j(Y, \mathcal{M}_0^\vee) = 0. \qquad (5.4.17)$$

Therefore

$$\begin{aligned}\chi(Y, M) &= \dim_{\mathbb{F}_q} H^0(Y, M) - \dim_{\mathbb{F}_q} H^1(Y, M) && (5.4.18)\\ &\geq \dim_k H^0(Y, \mathcal{M}_0^\vee) - \dim_k H^1(Y, \mathcal{M}_0^\vee) && (5.4.19)\\ &= \chi(Y, \mathcal{M}_0^\vee). && (5.4.20)\end{aligned}$$

The sheaf $\mathcal{M}_0^\vee$ is a locally-free $\mathcal{O}_X$-module of rank d. By the Riemann-Roch Theorem,

$$\chi(Y, \mathcal{M}_0^\vee) = d \cdot \chi(Y, \mathcal{O}_Y) + \deg \mathcal{M}_0^\vee \tag{5.4.21}$$

$$= d \cdot \chi(Y, \mathcal{O}_Y) - \deg \mathcal{M}_0 \tag{5.4.22}$$

$$= d \cdot \chi(Y, \mathcal{O}_Y) - \sum_{y \in Y(k)} \mathfrak{C}(M_{(x)}). \tag{5.4.23}$$

The proof is complete. □

5.5 Example: A rank-2 sheaf on the projective line

Suppose that $p \geq 5$. Let $E \to \operatorname{Spec} k$ be an elliptic curve with identity element $e \in |E|$. Choose sections $x, y \in \mathcal{O}_E(E \smallsetminus \{e\})$ so that

$$E - \{e\} \cong \operatorname{Spec} k[x,y]/(y^2 - x^3 - ax - b), \tag{5.5.1}$$

where $x^3 - ax - b$ is a polynomial with distinct roots. The inclusion

$$k[y] \hookrightarrow k[x,y]/(y^2 - x^3 - ax - b) \tag{5.5.2}$$

determines a degree-2 morphism

$$f \colon E \to \mathbb{P}^1. \tag{5.5.3}$$

There are 4 points of $\mathbb{P}^1$ at which f is ramified. Each point has ramification index 2. Let $a_1, a_2, a_3, a_4 \in \left|\mathbb{P}^1\right|$ be the points at which f is ramified, and let $b_1, b_2, b_3, b_4 \in |E|$ be their respective pre-images.

Let M be the pushforward of the constant sheaf $\underline{\mathbb{F}_q}$ via f. The closed stalks of M on $\mathbb{P}^1 \smallsetminus \{a_1, a_2, a_3, a_4\}$ are all 2-dimensional $\mathbb{F}_q$-vector spaces, and the closed stalks of M

on $\{a_1, a_2, a_3, a_4\}$ are all 1-dimensional $\mathbb{F}_q$-vector spaces. By Proposition 2.2.5,

$$\dim_{\mathbb{F}_q} H^0(\mathbb{P}^1, M) = \dim_{\mathbb{F}_q} H^0(E, \mathbb{F}_q) \tag{5.5.4}$$

and

$$\dim_{\mathbb{F}_q} H^1(\mathbb{P}^1, M) = \dim_{\mathbb{F}_q} H^1(E, \mathbb{F}_q). \tag{5.5.5}$$

The $\mathbb{F}_q$-dimension of $H^0(E, \mathbb{F}_q)$ is 1. The $\mathbb{F}_q$-dimension of $H^1(E, \mathbb{F}_q)$ is 0 if E supersingular, and 1 if E is ordinary. Thus the Euler characteristic of M is either 1 or 0.

For each $i \in \{1, 2, 3, 4\}$, let $A_i = \mathcal{O}_{\mathbb{P}^1, a_i}$ and $A_i' = \mathcal{O}_{E, b_i}$. Let K_i and K_i' denote the fraction fields of A_i and A_i', respectively. The stalk homomorphisms $A_i \to A_i'$ induce homomorphisms $K_i \hookrightarrow K_i'$ which are degree-2 Galois field extensions.

The localization $M_{(a_i)}$ is an $\mathbb{F}_q$-étale sheaf on Spec A_i of generic rank 2 which contains a trivial subsheaf of rank 1. The restriction of $M_{(a_i)}$ to Spec K_i' is trivial. The Galois group $\mathrm{Gal}(K_i'/K_i)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Since $p \neq 2$, the representation

$$\mathrm{Gal}(K_i'/K_i) \to \mathrm{GL}\left(M_{(a_i)}(\mathrm{Spec}\ K_i')\right) \tag{5.5.6}$$

can be decomposed into irreducible representations, thus inducing a decomposition of $M_{(a_i)}$. Let

$$M_{(a_i)} = M_i^+ \oplus M_i^-, \tag{5.5.7}$$

where M_i^+ is a trivial $\mathbb{F}_q$-sheaf of rank 1 and M_i^- is a nontrivial $\mathbb{F}_q$-sheaf of generic rank 1. The sheaves M_i^+ and M_i^- are the sheaves studied in Examples 4.5.1 and 4.5.4, respectively. By the computations in those examples, the minimal root index of

$$\mathrm{Hom}_{\mathbb{F}_q}\left(M_{(a_i)}, \mathcal{O}_{\mathrm{Spec}\ A}\right) = \mathrm{Hom}_{\mathbb{F}_q}\left(M_i^+, \mathcal{O}_{\mathrm{Spec}\ A}\right) \oplus \mathrm{Hom}_{\mathbb{F}_q}\left(M_i^-, \mathcal{O}_{\mathrm{Spec}\ A}\right) \tag{5.5.8}$$

is $0 + \frac{1}{2} = \frac{1}{2}$. Thus $\mathfrak{C}(M_{(a_i)}) = \frac{1}{2}$. Applying Theorem 5.4.9 to M yields

$$\chi(\mathbb{P}^1, M) \geq 2 \cdot \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) - \sum_{i=1}^{4} \mathfrak{C}(M_{(a_i)}) \quad (5.5.9)$$

$$= 2 \cdot 1 - 4 \cdot \frac{1}{2} \quad (5.5.10)$$

$$= 0. \quad (5.5.11)$$

This bound is an equality if and only if E is an ordinary elliptic curve.

Bibliography

[1] M. Emerton and M. Kisin. The Riemann-Hilbert correspondence for unit F-crystals. *Astérisque*, (293), 2004.

[2] E. Freitag and R. Kiehl. *Étale Cohomology and the Weil Conjecture*. Springer-Verlag, 1988.

[3] A. Grothendieck. *Groupes de Monodromie en Géométrie Algébrique (SGA 7)*. Number 288, 340 in Lecture Notes in Mathematics. Springer-Verlag, 1972-1973.

[4] A. Grothendieck. *Cohomologie ℓ-adique et Fonctions L (SGA 5)*. Number 589 in Lecture Notes in Mathematics. Springer-Verlag, 1977.

[5] S. J. Edixhoven, B. J. J. Moonen, and F. Oort. Open problems in algebraic geometry. *Bulletin des Sciences Mathématiques*, 125(1):1–22, 2001.

[6] R. Pink. Euler-Poincaré formula in equal characteristic under ordinariness assumptions. *Manuscripta Mathematica*, 102(1):1–24, 2000.